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Towards effective shell modelling with the FEniCS project

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Outline

- Introduction
- Shells:
  - chart
  - shear-membrane-bending and membrane-bending models
  - example forms
- Two proposals for discussion:
  - geometry: chart object for describing shell geometry
  - discretisation: projection/reduction operators for implementation of generalised displacement methods
- Summary

So far...

- dolfin manifold support already underway, merged into trunk [Marie Rognes, David Ham, Colin Cotter]
- I have already implemented locking-free (uncurved) beams and plate structures using dolfin manifold
- Next step: curved surfaces, generalised displacement methods (?)
- Aim of my talk is to start discussion on the best path

Why shells?

- *The mathematics:* Shells are three-dimensional elastic bodies which occupy a 'thin' region around a two-dimensional manifold situated in three-dimensional space
- *The practical advantages:* Shell structures can hold huge applied loads over large areas using a relatively small amount of material. Therefore they are used abundantly in almost all areas of mechanical, civil and aeronautical engineering.
- *The computational advantages:* A three-dimensional problem is reduced to a two-dimensional problem. Quantities of engineering relevance are computed directly.

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Figure : British Museum Great Court. Source: Wikimedia Commons.

Figure : Specialized OSBB bottom bracket. Source: bikeradar.com

A huge field

There are *many* different ways of:

- obtaining shell models
- representing the geometry of surfaces on computers
- discretising shell models successfully

And therefore we need suitable *abstractions* to ensure generality and extensibility of any shell modelling capabilities in FEniCS.

Two methodologies

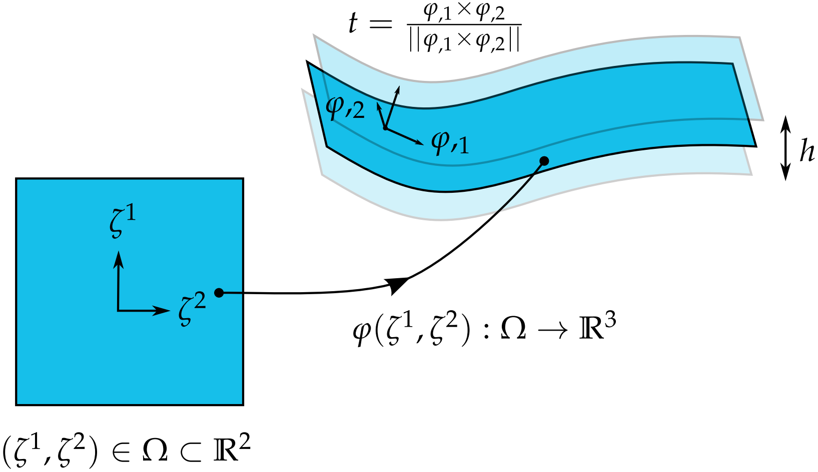
Mathematical Model approach:

1. Derive a mathematical shell model.
2. Discretise that model using appropriate numerical method for description of geometry and fields.

Degenerated Solid approach:

1. Begin with a general 3D variational formulation for the shell body.
2. Degenerate a solid 3D element by inferring appropriate FE interpolation at a number of discrete points.
3. No explicit mathematical shell model, one may be implied.

Mathematical model



Mathematical model

shear-membrane-bending (smb) model

Find $U \in \mathcal{V}_{smb}$:

$$h^3 A_b(U,V) + h A_s(U,V) + h A_m(U,V) = F(V) \quad \forall V \in \mathcal{V}_{smb} \quad (1)$$

membrane-bending (mb) model

Find $U \in \mathcal{V}_{mb}$:

$$h^3 A_b(U,V) + h A_m(U,V) = F(V) \quad \forall V \in \mathcal{V}_{mb} \quad (2)$$

Discretisation

smb models vs mb models

- smb model takes into account the effects of shear; ‘closer’ to the 3D solution for thick shells, matches the mb model for thin shells.
- Boundary conditions are better represented in smb model; hard and soft supports, boundary layers.
- smb $U \in H^1(\Omega)$ vs mb $U \in H^2(\Omega)$

Mathematical model

Let's just take a look at the bending bilinear form A_b for the mb model:

$$A_b(U,V) = \int_\Omega \rho_{\alpha\beta} H_b^{\alpha\beta\gamma\delta} \rho_{\gamma\delta} \, dA \tag{3a}$$

$$\begin{aligned} \rho_{\alpha\beta} := \varphi_{,\alpha\beta} \cdot t \frac{1}{j}(u_{,1} \cdot (\varphi_{,2} \times t) - u_{,2} \cdot (\varphi_{,1} \times t)) \\ + \frac{1}{j}(u_{,1} \cdot (\varphi_{,\alpha\beta} \times \varphi_{,2} - u_{,2} \cdot (\varphi_{,\alpha\beta} \times \varphi_{,1})) \\ - u_{,\alpha\beta} \cdot t \end{aligned} \tag{3b}$$

$$H_b^{\alpha\beta\gamma\delta} := \frac{Eh^3}{12(1-\nu^2)} \left(\nu (\varphi'^{\alpha} \cdot \varphi'^{\beta}) + \dots \right) \tag{3c}$$

Geometry

Continuous model

Terms describing the differential geometry of the shell mid-surface. The mid-surface is defined by the chart function.

Discrete model

We do not (usually) have an explicit representation of the chart. It must be constructed implicitly from the mesh and/or data from a CAD model. There are many different ways of doing this.

Geometry

Proposal 1

A base class `Chart` object which exposes various new symbols describing the geometry of the shell surface. Specific subclasses of `Chart` will implement a particular computational geometry procedure. The user can then express their mathematical shell model independently from the underlying geometrical procedure using the provided high-level symbols.

```
shell_mesh = mesh("shell.xml")
normals = MeshFunction(...)
C = FunctionSpace(shell_mesh, "CG", 2)

chart = Chart(shell_mesh, C,
    method="patch_averaged")
chart = Chart(shell_mesh, C,
    method="CAD_normals", normals=normals)
...
b_cnt = chart.contravariant_basis()
b_cov = chart.covariant_basis()
dA = chart.measure()
a = chart.first_fundamental_form()
...
A_b = ...
```

Current discretisation options

mb model:

- $H^2(\Omega)$ conforming finite elements
- DG methods

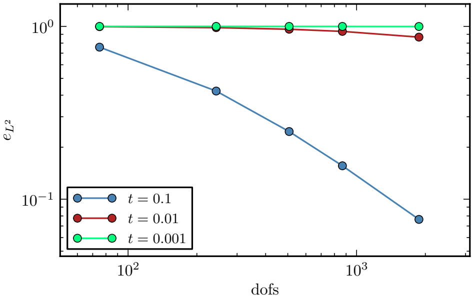
smb model:

- straight $H^1(\Omega)$ conforming finite elements
- mixed finite elements (CG, DG)
- generalised displacement methods

A quick note

For simplicity I will just talk about the smb model reduced to plates, the chart function is the identity matrix; considerably simpler asymptotic behaviour but concepts apply to shells also.

Locking



Locking

Inability of the basis functions to represent the limiting Kirchhoff mode.

Move to a mixed formulation

Treat the shear stress as an *independent* variational quantity:

$$\gamma_h = \lambda \bar{t}^{-2} (\nabla z_{3h} - \boldsymbol{\theta}_h) \in \mathcal{S}_h$$

Discrete Mixed Weak Form

Find $(z_{3h}, \boldsymbol{\theta}_h, \gamma_h) \in (\mathcal{V}_{3h}, \mathcal{R}_h, \mathcal{S}_h)$ such that for all $(y_{3h}, \boldsymbol{\eta}, \boldsymbol{\psi}) \in (\mathcal{V}_{3h}, \mathcal{R}_h, \mathcal{S}_h)$:

$$a_b(\boldsymbol{\theta}_h; \boldsymbol{\eta}) + (\gamma_h; \nabla y_3 - \boldsymbol{\eta})_{L^2} = f(y_3)$$

$$(\nabla z_{3h} - \boldsymbol{\theta}_h; \boldsymbol{\psi})_{L^2} - \frac{\bar{t}^2}{\lambda} (\gamma_h; \boldsymbol{\psi})_{L^2} = 0$$

Move back to a displacement formulation

Linear algebra level: Eliminate the shear stress unknowns *a priori* to solution

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{Bmatrix} u \\ \gamma \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \tag{5}$$

To do this we can rearrange the second equation and then if and only if C is diagonal/block-diagonal we can invert cheaply giving a problem in original displacement unknowns:

$$(A + BC^{-1}B^T)u = f \tag{6}$$

Currently, this can be done with CBC.Block TRIA0220, TRIA1B20 [Arnold and Brezzi][Boffi and Lovadina]

https://answers.launchpad.net/dolfin/+question/143195 David Ham, Kent Andre-Mardal, Anders Logg, Joachim Haga and myself

```
A, B, BT, C = [assemble(a), assemble(b),
    assemble(bt), assemble(c)]
K = collapse(A - B * LumpedInvDiag(C) * BT)
```

Move back to a displacement formulation

Variational form level:

$$\gamma_h = \lambda \bar{t}^{-2} \Pi_h (\nabla z_{3h} - \theta_h) \tag{7}$$

$$\Pi_h : \mathcal{V}_{3h} \times \mathcal{R}_h \to \mathcal{S}_h \tag{8}$$

giving:

$$a_b(\boldsymbol{\theta}_h; \boldsymbol{\eta}) + (\Pi_h(\nabla z_3 - \theta); \nabla y_3 - \boldsymbol{\eta})_{L^2} = f(y_3) \tag{9}$$

Proposal 2

A new class `Projection` in UFL that signals to FFC that a projection between `FunctionSpace` objects is required. This requires additions in DOLFIN, UFL, FFC and FIAT.

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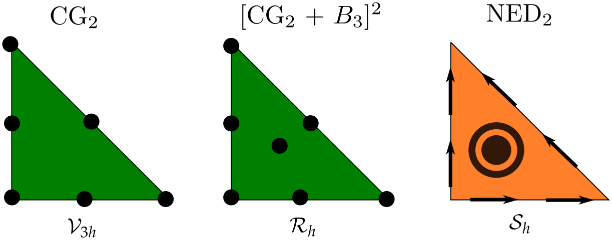

```
...
V_3 = FunctionSpace(mesh, "CG", 2)
R = VectorFunctionSpace(mesh, "CG", 2, dim=2) +
     VectorFunctionSpace(mesh,"B", 3, dim=2)
S = FunctionSpace(mesh, "N1curl", order=2)
Pi_h = Projection(from=R, to=S)
...
U = MixedFunctionSpace([R, V_3])
theta, z_3 = TrialFunctions(U)
eta, y_3 = TestFunctions(U)
a_s = inner(grad(z_3) - Pi_h(theta), grad(y_3)
    - Pi_h(eta))*dx
```

Summary

- A big field with lots of approaches; need appropriate abstractions (inc. other PDEs on surfaces)
- Proposal 1: Expression of geometric terms in shell models using a natural form language which reflects the underlying mathematics
- Proposal 2: Effective discretisation options for the implementation of generalised displacement methods

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Thanks for listening.