The Periodic Table of Finite Elements

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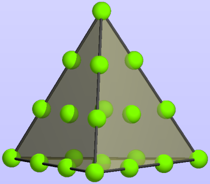
FEniCS 2013

The Lagrange finite element spaces, $\mathcal{P}_r(\mathcal{T}_h)$

- Elements: A triangulation $\mathcal{T}_h$ consisting of simplices T
- Shape functions: $V(T) = \mathcal{P}_r(T)$, some $r \geq 1$
- Degrees of freedom (which must be *unisolvent*):
  - $v \in \Delta_0(T)$: $u \mapsto u(v)$
  - $e \in \Delta_1(T)$: $u \mapsto \int_e (\mathrm{tr}_e\, u) q, \quad q \in \mathcal{P}_{r-2}(e)$
  - $f \in \Delta_2(T)$: $u \mapsto \int_f (\mathrm{tr}_f\, u) q, \quad q \in \mathcal{P}_{r-3}(f)$
  - T: $u \mapsto \int_T u\, q, \quad q \in \mathcal{P}_{r-4}(T)$



For a general simplex of any dimension and a face f of any dimension:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) q, \quad q \in \mathcal{P}_{r-d-1}(f),\ f \in \Delta_d(T),\ d \geq 0$$

Assembled piecewise polynomials are continuous, and

$$\mathcal{P}_r(\mathcal{T}_h) = \{\, u \in H^1(\Omega) \mid u|_T \in V(T)\, \forall T \in \mathcal{T}_h \,\}$$

The Maxwell eigenvalue problem with Lagrange elements

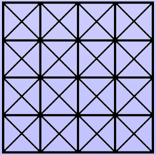
Find nonzero $u \in H(\mathrm{curl})$ such that

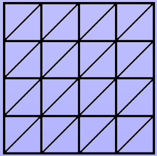
Boffi–Gastaldi

$$\int \mathrm{curl}\, u \cdot \mathrm{curl}\, v \, dx = \lambda \int u \cdot v \, dx, \quad \forall v \in H(\mathrm{curl})$$

$$\Omega = (0,\pi) \times (0,\pi), \quad \lambda = m^2 + n^2, \quad m, n > 0$$

elts: 16	64	256	1024	4096	
2.2606	2.0679	2.0171	2.0043	2.0011	
4.8634	5.4030	5.1064	5.0267	5.0067	
5.6530	5.4030	5.1064	5.0267	5.0067	
5.6530	5.6798	5.9230	5.9807	5.9952	!!
11.3480	9.0035	8.2715	8.0685	8.0171	
1.3488	0.2576	0.0587	0.0143	0.0036	
1.5349	0.4196	0.0896	0.0214	0.0053	
2.4756	0.9524	0.1805	0.0417	0.0102	
5.5582	1.4513	0.2938	0.0686	0.0169	
5.7592	1.7446	0.3694	0.0826	0.0200	

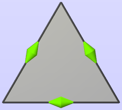




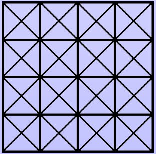
The Maxwell eigenvalue problem with $H(\mathrm{curl})$ elements

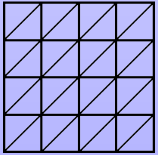
```
#V = VectorFunctionSpace(mesh, "Lagrange", 1)
V = FunctionSpace(mesh, "N1curl", 1)
```

Shape fns: $(a - bx_2, c + bx_1)$ DOFs: $u \mapsto \int_e u \cdot t$



elts: 16	64	256	1024	4096
1.8577	1.9655	1.9914	1.9979	1.9995
4.1577	4.8929	4.9749	4.9938	4.9985
4.1577	4.8929	4.9749	4.9938	4.9985
8.2543	7.4306	7.8619	7.9657	7.9914
9.7268	9.8498	9.9858	9.9975	9.9994
2.1098	2.0324	2.0084	2.0021	2.0005
3.5416	4.8340	4.9640	4.9912	4.9978
4.8634	5.0962	5.0259	5.0066	5.0017
9.7268	8.0766	8.1185	8.0332	8.0085
9.7268	8.9573	9.7979	9.9506	9.9877





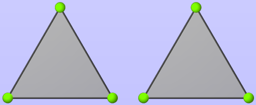
A good element for this problem in both theory *and* practice. . .

Darcy flow

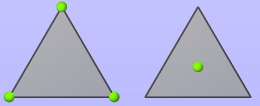
$$u = \frac{k}{\mu}\,\mathrm{grad}\,p, \qquad \mathrm{div}\,u = f$$

Find $(u, p) \in H(\mathrm{div}) \times L^2$ such that

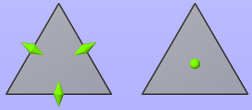
$$\int \left(\frac{\mu}{k}\, u \cdot v - p\,\mathrm{div}\,v + \mathrm{div}\,u\,q \right)\,dx = \int f\,q\,dx, \quad \forall (v, q) \in H(div) \times L^2$$



Lagrange–Lagrange is singular

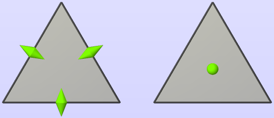


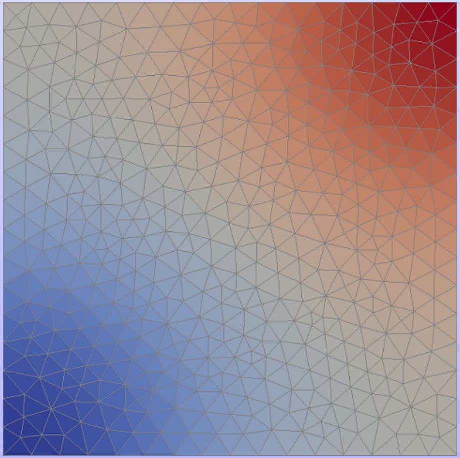
Lagrange–DG is unstable in > 1 dimensions

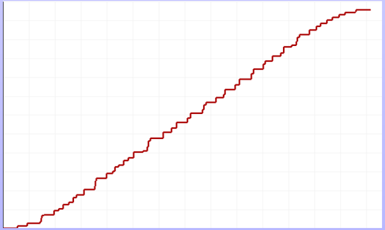


RT–DG is stable and convergent

Darcy flow computed with RT–DG

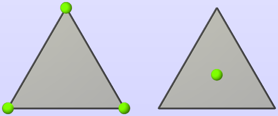


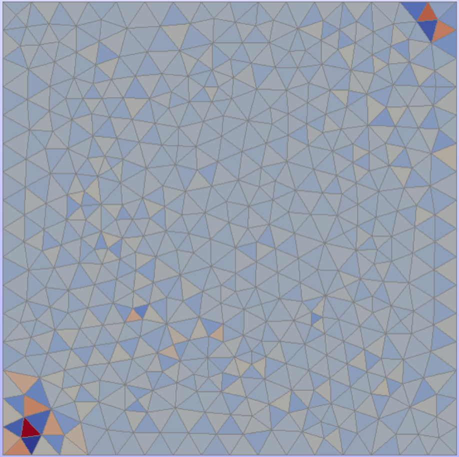


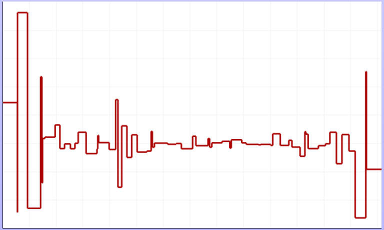


pressure field

Darcy flow computed with Lagrange–DG

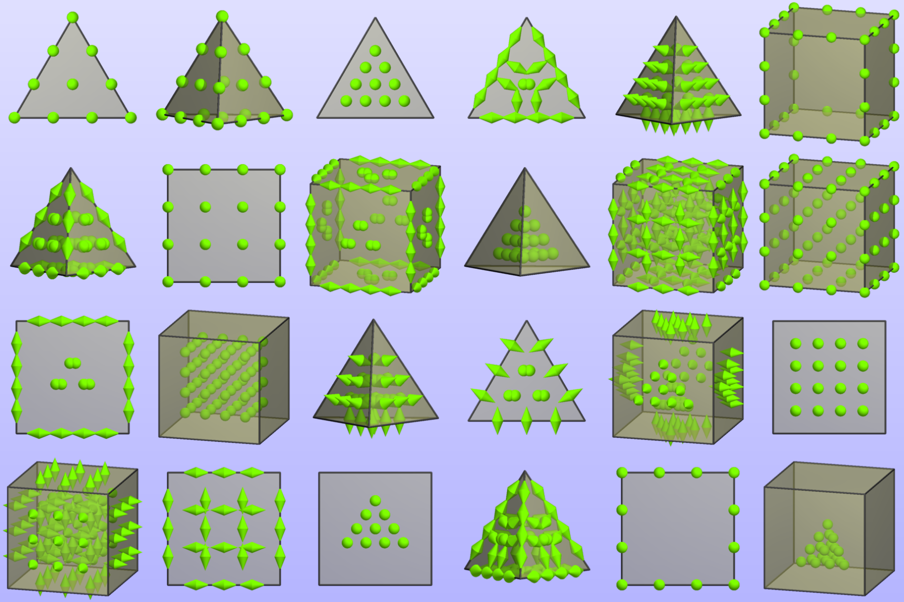






pressure field

The Finite Element Zoo (Cubic Pavillion)



The Finite Element Exterior Calculus Viewpoint

Differential forms and the L^2 de Rham complex

- Differential k-forms, $\Lambda^k(\Omega)$: defined for any manifold Ω, $0 \le k \le \dim\Omega$
- 0-forms are simply functions $\Omega \to \mathbb{R}$ and 1-forms are covector fields. In local coordinates, the general k-form is
$$u = \sum_\sigma f_\sigma\, dx^\sigma := \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} f_{\sigma_1\cdots\sigma_k}\, dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k}$$
- The wedge product of a k-form and an l-form is a $(k+l)$-form.
- The exterior derivative du of a k-form is a $(k+1)$-form
- A k-form can be integrated over a k-dimensional subset of Ω
- $F : \Omega \to \Omega'$ induces a pullback F^* taking k-forms on Ω' to k-forms on Ω
- The pullback of the inclusion is the trace.
- Stokes theorem: $\displaystyle\int_\Omega du = \int_{\partial\Omega} \operatorname{tr} u, \quad u \in \Lambda^{k-1}(\Omega)$

On a *Riemannian* manifold, the space $L^2\Lambda^k(\Omega)$ is defined, leading to
$$H\Lambda^k(\Omega) = \{\, u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \,\}$$

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^{n-1}(\Omega) \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

Differential forms in $\mathbb{R}^3$ and the PDEs of math physics

Ω a domain in $\mathbb{R}^3$

$$0 \longrightarrow H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} H\Lambda^2(\Omega) \xrightarrow{d} H\Lambda^3(\Omega) \longrightarrow 0$$

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0$$

0-forms:	temperature; electric potential; displacement
1-forms:	temperature gradient; electric field; magnetic field; strain
2-forms:	heat flux; magnetic flux; vorticity; stress
3-forms:	charge density; mass density; load

> "Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area."
>
> – James Clerk Maxwell, *Treatise on Electricity & Magnetism*, 1891

Finite Element Exterior Calculus

FEEC identifies the properties that finite element subspaces of $H\Lambda^k$ should possess:

The finite element spaces should form a *subcomplex* of the de Rham complex, and the projections induced by the degrees of freedom should *commute* with the exterior derivative.

$$\begin{array}{ccccccccccc}
0 & \to & H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) & \to & 0 \\
 & & \pi_h^0 \big\downarrow & & \pi_h^1 \big\downarrow & & & & \pi_h^2 \big\downarrow & & \\
0 & \to & \Lambda^0(\mathcal{T}_h) & \xrightarrow{d} & \Lambda^1(\mathcal{T}_h) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda^n(\mathcal{T}_h) & \to & 0
\end{array}$$

DNA-Falk-Winther:

Finite element exterior calculus, homological techniques and applications, Acta Numer ’06

Finite element exterior calculus: from Hodge theory to numerical stability, BAMS ’10

Simplicial elements

The $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ families of elements in $\mathbb{R}^n$

- Triangulation $\mathcal{T}_h$ consists of n-simplices T
- Shape functions: $V(T) = \mathcal{P}_r\Lambda^k(T)$ or $\mathcal{P}_r^-\Lambda^k(T)$
- DOFs?

$\mathcal{P}_r^-\Lambda^k(T)$ is defined via the *Koszul differential* κ:

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(T)$$

- $\kappa : \Lambda^k \to \Lambda^{k-1}$, $\kappa(dx^i) = x^i$, $\kappa(u \wedge v) = (\kappa u)\wedge v + (-)^k u \wedge (\kappa v)$
- $\kappa(f\, dx^{\sigma_1}\wedge \cdots \wedge dx^{\sigma^k}) = \sum_{i=1}^k (-)^i f\, x^{\sigma_i}\, dx^{\sigma_1}\wedge \cdots \widehat{dx^{\sigma_i}} \cdots \wedge dx^{\sigma^k}$
- In $\mathbb{R}^3$: $\quad \mathcal{P}_r\Lambda^3 \xrightarrow{\boldsymbol{x}} \mathcal{P}_{r+1}\Lambda^2 \xrightarrow{\times\boldsymbol{x}} \mathcal{P}_{r+2}\Lambda^1 \xrightarrow{\cdot\boldsymbol{x}} \mathcal{P}_{r+3}\Lambda^0$
- $\kappa \circ \kappa = 0$
- *Homotopy property:*

$$(d\kappa + \kappa d)u = (r+k)u \quad \text{if } u \in \mathcal{P}_r\Lambda^k \text{ is homogeneous}$$

e.g., $\quad \operatorname{curl}(\vec{x} \times \vec{v}) + \vec{x}\,(\operatorname{div}\vec{v}) = (\deg\vec{v} + 2)\,\vec{v}$

Some consequences of the homotopy formula

$(d\kappa + \kappa d)u = cu$

1) $c\,\kappa u = \kappa d\kappa u.$ Therefore, $d\kappa u = 0 \implies \kappa u = 0$

Thus, if $u \in \mathcal{P}_r^- \Lambda^k$ and $du = 0$, then $u \in \mathcal{P}_{r-1}\Lambda^k$.

2) The polynomial de Rham complex

$$0 \to \mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n \to 0$$

and the Koszul complex

$$0 \leftarrow \mathcal{H}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{H}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{H}_{r-n}\Lambda^n \leftarrow 0$$

are exact.

3) From this we can compute the dimension of $\kappa\mathcal{H}_r\Lambda^k$,
and so of $\mathcal{P}_r^-\Lambda^k$:

$$\dim \mathcal{P}_r^-\Lambda^k = \binom{r+n}{r+k}\binom{r+k-1}{k} \quad \text{cf. } \dim \mathcal{P}_r\Lambda^k = \binom{r+n}{r+k}\binom{r+k}{k}$$

Characterization of the $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ spaces

Theorem

The following spaces of polynomial differential k-forms are invariant under all affine transformations of $\mathbb{R}^n$:

- $\mathcal{P}_r\Lambda^k, \quad r \geq 0,$
- $\mathcal{P}_r^-\Lambda^k, \quad r \geq 1,$
- $\{\, u \in \mathcal{P}_r\Lambda^k \mid du \in \mathcal{P}_s\Lambda^k \,\}, \quad r \geq 1, s < r - 1$

Moreover, these are the only affine invariant proper subspaces.

The proof is based on the representation theory of $GL(n)$.

Degrees of freedom

DOFs for $\mathcal{P}_r\Lambda^k(T)$ (DNA-Falk-Winther '06):

$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f),\ f \in \Delta_d(T),\ d = \dim f \geq k$$

DOFs for $\mathcal{P}^-_r\Lambda^k(T)$ (Hiptmair '99):

$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f),\ f \in \Delta_d(T),\ d = \dim f \geq k$$

- Continuity is exactly that of $H\Lambda^k = \{\, u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \,\}$

$$\mathcal{P}_r\Lambda^k(\mathcal{T}_h) = \{\, u \in H\Lambda^k \mid u|_T \in \mathcal{P}_r\Lambda^k(T),\ \forall T \in \mathcal{T}_h \,\}.$$

or $\mathcal{P}^-_r$

- The spaces form subcomplexes with commuting projections:

$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}_h) \to 0$$

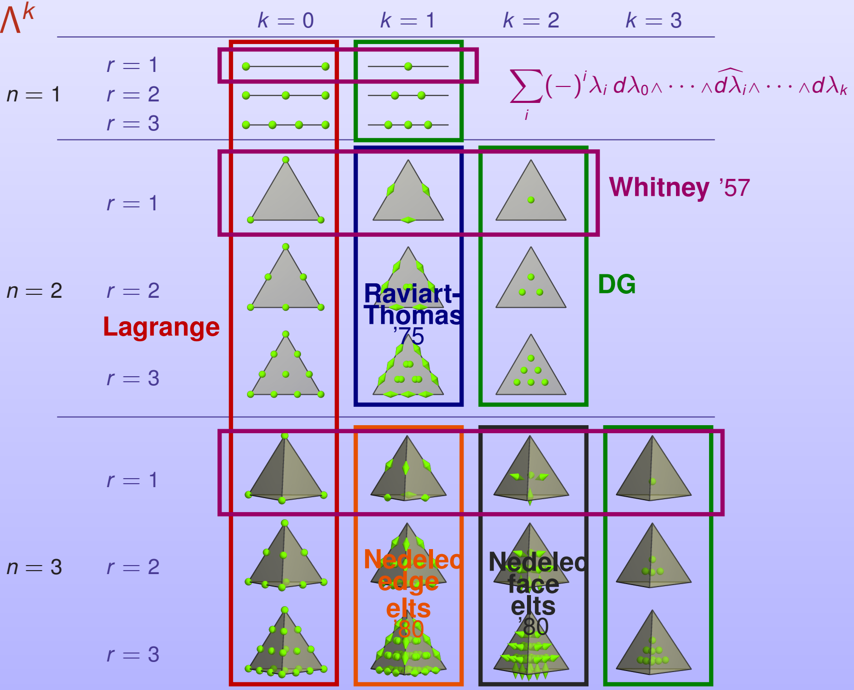
$$0 \to \mathcal{P}^-_r\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}^-_r\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}^-_r\Lambda^n(\mathcal{T}_h) \to 0$$

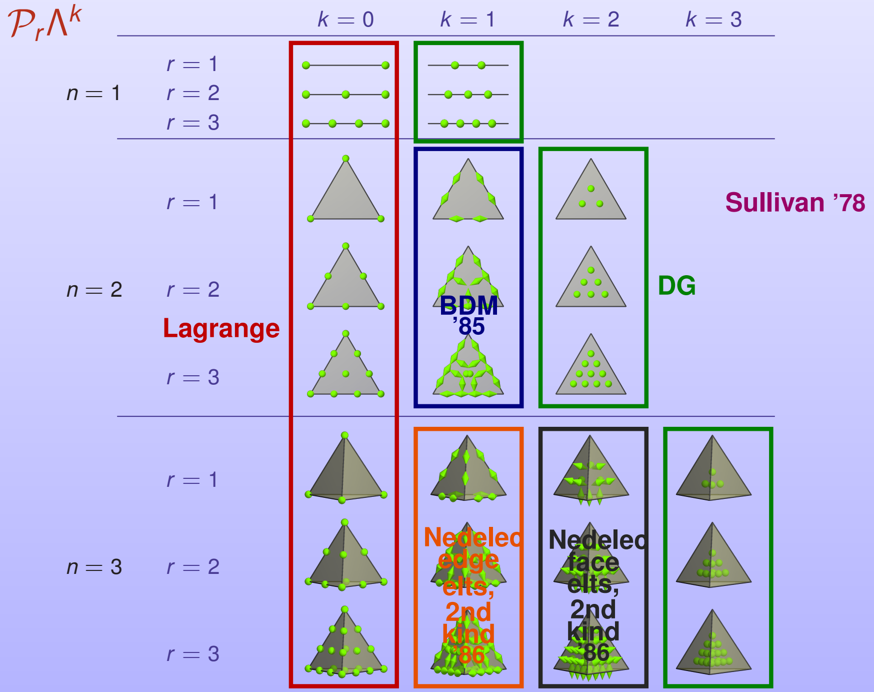
decreasing degree

constant degree

Unisolvence?

$\mathcal{P}_r^-\Lambda^k$





$\mathcal{P}_r\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange
BDM '85
DG
Sullivan '78
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86

FEniCS syntax

FEniCS supports all the $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ spaces in 1, 2, and 3 dimensions.

```
V = FunctionSpace(mesh, "P- Lambda", r, k)

V = FunctionSpace(mesh, "P Lambda", r, k)
```

These are synonyms for the more traditional names.

Unisolvence

Unisolvence for Lagrange elements in n dimensions

Shape fns: $\mathcal{P}_r(T)$, DOFs: $u \mapsto \int_f (\operatorname{tr}_f u) q, \ q \in \mathcal{P}_{r-d-1}(f), \ d = \dim f$

DOF count:

$\#\Delta_d(T)$ $\dim \mathcal{P}_{r-d-1}(f_d)$ $\dim \mathcal{P}_r(T)$

$$\#\mathrm{DOF} = \sum_{d=0}^{n} \binom{n+1}{d+1} \binom{r-1}{d} = \binom{r+n}{n} = \dim \mathcal{P}_r(T).$$

Unisolvence proved by induction on dimension.

Suppose $u \in \mathcal{P}_r(T)$ and all DOFs vanish. Let f be a face of T. Note

- $\operatorname{tr}_f u \in \mathcal{P}_r(f)$, so is a Lagrange shape function on the face
- all the Lagrange DOFs on the face applied to $\operatorname{tr}_f u$ are DOFs on T applied to u, so vanish

Therefore $\operatorname{tr}_f u$ vanishes by the inductive hypothesis. Thus $u \in \mathring{\mathcal{P}}_r(T) \implies$

$$u = \left(\prod_{i=0}^{n} \lambda_i\right) p, \quad p \in \mathcal{P}_{r-n-1}(T)$$

The explicit choice of weight fn $q = p$ in the interior DOFs implies $p = 0$.

Steps to verifying unisolvence

1. Verify that the number of DOFs equals $\dim V(T)$

2. Verify the *trace properties:*
   a) $\mathrm{tr}_f\, V(T) \subset V(f)$, and
   b) the pullback $\mathrm{tr}_f^* : V(f)^* \to V(T)^*$ takes DOFs for $V(f)$ to DOFs for $V(T)$

3. $u \in \mathring{V}(T)$ & the interior DOFs vanish $\implies$ $u = 0$

subspace w/ vanishing trace

1,2,3 $\implies$ unisolvence, by induction on dimension

Unisolvence for $\mathcal{P}_r^-\Lambda^k$

1. $\dim \mathcal{P}_r^-\Lambda^k(T) = \binom{r+n}{r+k}\binom{r+k-1}{k}$ (homotopy property)

#DOFs $= \sum_{d\ge k} \#\Delta_d(T) \dim \mathcal{P}_{r+k-d-1}\Lambda^k(\mathbb{R}^d) = \sum_{d\ge k}\binom{n+1}{d+1}\binom{r+k-1}{d}\binom{d}{k}$

These are equal by elementary manipulations.

2. The trace property follows from definitions (since κ commutes with tr_f).

3. So we only need show:

$(\dagger)\ u \in \mathring{\mathcal{P}}_r^-\Lambda^k(T)$ & $(*)\ \int_T u\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T) \implies u = 0$

A weaker result can be proven by an *explicit choice of q*

$(\ddagger)\ u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T)$ & $(*) \implies u = 0$

So we only need to show that $u \in \mathcal{P}_{r-1}\Lambda^k(T)$.

By the homotopy formula, $u \in \mathcal{P}_r^-\Lambda^k$, $du = 0 \implies u \in \mathcal{P}_{r-1}\Lambda^k$, so it suffices to show that $du = 0$.

But $du \in \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T)$ so satisfies $(\ddagger)$ with $k \to k+1$. The hypothesis $(*)$ for du then becomes: $(*)\ \int_T du\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n}\Lambda^{n-k-1}(T)$ which holds by integration by parts and $(*)$.

Summary for simplicial elements

The argument adapts easily to $\mathcal{P}_r\Lambda^k$. Thus a single argument proves unisolvence for all of the most important simplicial FE spaces at once.

To obtain the "best" proof, it is necessary

- to consider $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ together
- to consider all form degrees k
- to consider general dimension n

"A finite element which does not work in *n*-dimensions is probably not so good in 2 or 3 dimensions."

Cubical elements

The tensor product construction

DNA–Boffi–Bonizzoni 2012

Suppose we have a de Rham subcomplex V on an element $S \subset \mathbb{R}^m$:

$$\cdots \to V^k \xrightarrow{d} V^{k+1} \to \cdots \qquad V^k \subset \Lambda^k(S)$$

and another, W, on another element $T \subset \mathbb{R}^n$:

$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

Shape fns: $$(V \wedge W)^k = \bigoplus_{i+j=k} \pi_S^* V^i \wedge \pi_T^* W^j \qquad (\pi_S : S \times T \to S)$$

DOFs: $$(\eta \wedge \rho)(\pi_S^* v \wedge \pi_T^* w) := \eta(v)\rho(w)$$

Finite element differential forms on cubes: the $\mathcal{Q}_r^- \Lambda^k$ family

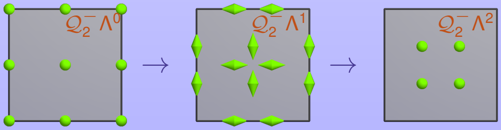
Start with the simple 1-D degree r finite element de Rham complex, V_r:

$$0 \to \mathcal{P}_r\Lambda^0(I) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(I) \to 0$$

$$u(x) \quad \to \quad u'(x)\,dx$$

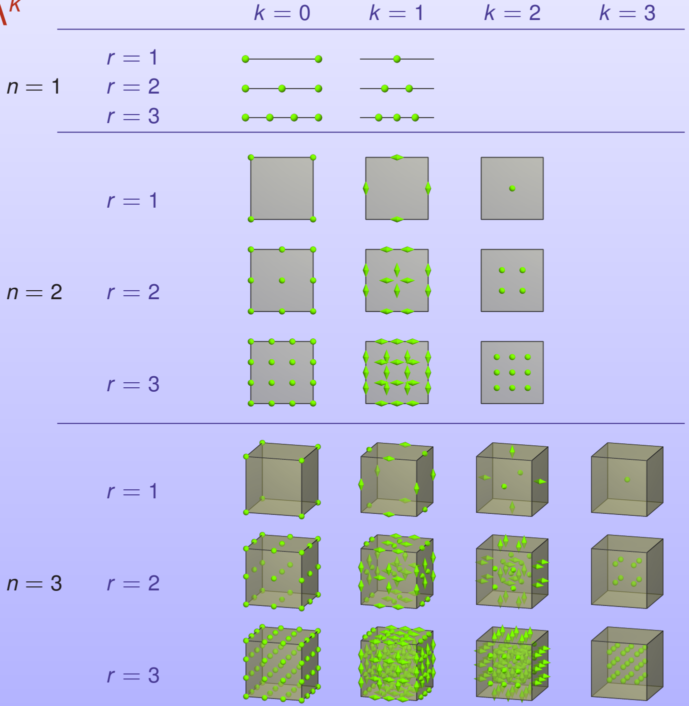
Take tensor product n times: $\mathcal{Q}_r^- \Lambda^k(I^n) := (V_r \wedge \cdots \wedge V_r)^k$

$\mathcal{Q}_r^- \Lambda^0 = \mathcal{Q}_r,$
$\mathcal{Q}_r^- \Lambda^1 = \mathcal{Q}_{r-1,r,r,\ldots} dx^1 + \mathcal{Q}_{r,r-1,r,\ldots} dx^2 + \cdots,$
$\mathcal{Q}_r^- \Lambda^2 = \mathcal{Q}_{r-1,r-1,r,\ldots} dx^1 \wedge dx^2 + \cdots, \quad \ldots$



constant degree

$\mathcal{Q}_r^-\Lambda^k$



The 2nd family on cubes: 0-forms

DNA–Awanou 2011

The $\mathcal{Q}_r^- \Lambda^k$ family reduces to $\mathcal{Q}_r$ when $k = 0$. For the second family, we get the serendipy space $\mathcal{S}_r$.

2-D shape fns: $\mathcal{S}_r(I^2) = \mathcal{P}_r(I^2) \oplus \text{span}[x_1^r x_2, x_1 x_2^r]$

DOFs: $u \mapsto \int_f \text{tr}_f\, u\, q, \quad q \in \mathcal{P}_{r-2d}(f), f \in \Delta(I^n)$

n-D shape fns: $\mathcal{S}_r(I^n) = \mathcal{P}_r(I^n) \oplus \bigoplus_{\ell \geq 1} \mathcal{H}_{r+\ell,\ell}(I^n)$

$\mathcal{H}_{r,\ell}(I^n) =$ span of monomials of degree r, linear in $\geq \ell$ variables

The 2nd family of finite element differential forms on cubes

DNA–Awanou 2012

The $\mathcal{S}_r\Lambda^k(I^n)$ family of FEDFs, uses the serendipity spaces for 0-forms, and serendipity-like DOFs.

DOFs: $u \mapsto \int_f \operatorname{tr}_f u \wedge q, \quad q \in \mathcal{P}_{r-2d}\Lambda^{d-k}(f),\ f \in \Delta(I^n)$

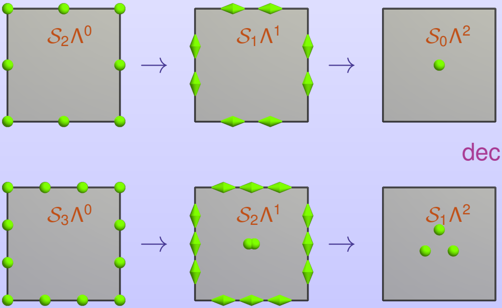
Shape fns:

$$\mathcal{S}_r\Lambda^k(I^n) = \mathcal{P}_r\Lambda^k(I^n) \oplus \bigoplus_{\ell\geq 1} \underbrace{[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1}(I^n) \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k(I^n)]}_{\deg = r+\ell}$$

$$\mathcal{H}_{r,\ell}\Lambda^k(I^n) = \text{ span of monomials } x_1^{\alpha_i}\cdots x_n^{\alpha_n}\, dx_{\sigma_1}\wedge\cdots\wedge dx_{\sigma_k},$$
$$|\alpha| = r, \text{linear in} \geq \ell \text{ variables not counting the } x_{\sigma_i}$$

These spaces satisfy the trace property, and unisolvence holds for all $n \geq 1$, $r \geq 1$, $0 \leq k \leq n$.

The 2nd cubic family in 2-D



decreasing degree

$\mathcal{S}_r\Lambda^k(I^2)$

k	1	2	3	4	5
0	4	8	12	17	23
1	8	14	22	32	44
2	3	6	10	15	21

$\mathcal{Q}_r^-\Lambda^k(I^2)$

k	1	2	3	4	5
0	4	9	16	25	36
1	4	12	24	40	60
2	1	4	9	16	25

The 3D shape functions in traditional FE language

$\mathcal{S}_r\Lambda^0$: polynomials u such that $\deg u \le r + \operatorname{ldeg} u$

$\mathcal{S}_r\Lambda^1$:

$$(v_1, v_2, v_3) + (x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)) + \operatorname{grad} u,$$

$v_i \in \mathcal{P}_r$, $w_i \in \mathcal{P}_{r-1}$ independent of x_i, $\deg u \le r + \operatorname{ldeg} u + 1$

$\mathcal{S}_r\Lambda^2$:

$$(v_1, v_2, v_3) + \operatorname{curl}(x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)),$$

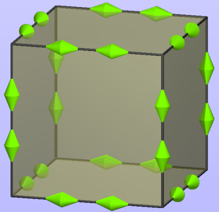
$v_i, w_i \in \mathcal{P}_r(I^3)$ with w_i independent of x_i

$\mathcal{S}_r\Lambda^3$: $v \in \mathcal{P}_r$

Dimensions and low order cases

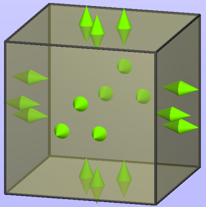
	$\mathcal{S}_r\Lambda^k(I^3)$				
k	1	2	3	4	5
0	8	20	32	50	74
1	24	48	84	135	204
2	18	39	72	120	186
3	4	10	20	35	56

	$\mathcal{Q}_r^-\Lambda^k(I^3)$				
k	1	2	3	4	5
0	8	27	64	125	216
1	12	54	96	200	540
2	6	36	108	240	450
3	1	8	27	64	125



$\mathcal{S}_1\Lambda^1(I^3)$

new element



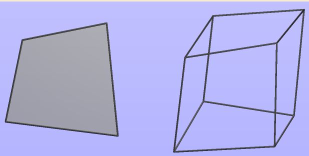
$\mathcal{S}_1\Lambda^2(I^3)$

corrected element

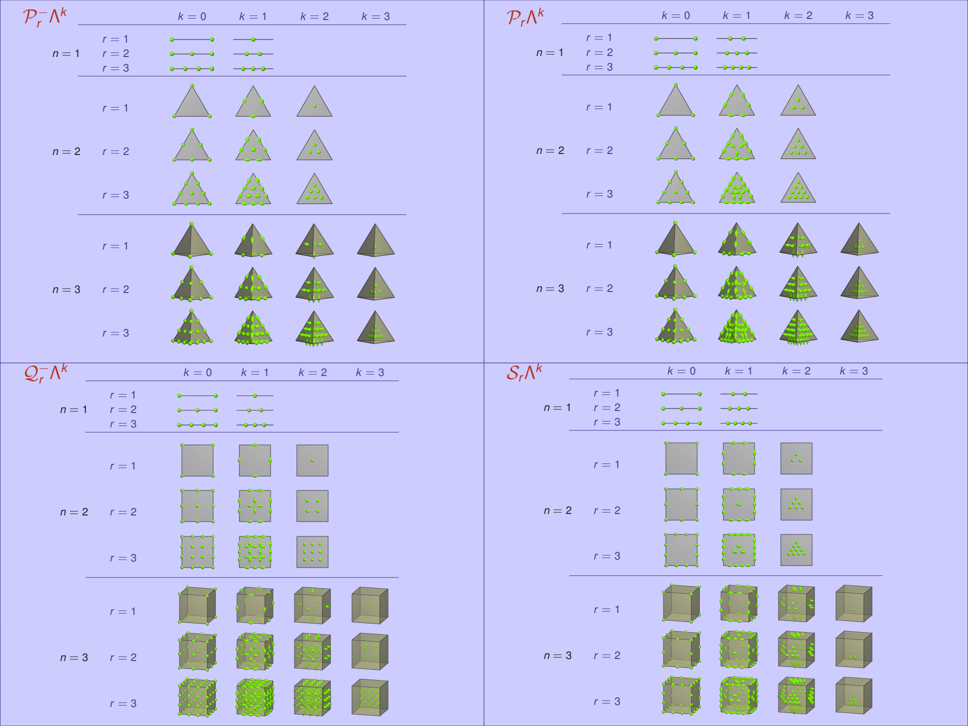
Approximation properties

On cubes the $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r^-\Lambda^k$ spaces provide the expected order of approximation. Same is true on parallelotopes, but accuracy is lost by non-affine distortions, *with greater loss, the greater the form degree k*.

- The L^2 approximation rate of the space $\mathcal{Q}_r = \mathcal{Q}_r^-\Lambda^0$ is $r+1$ on either affinely or multilinearly mapped elements.
- The rate for $\mathcal{S}_r = \mathcal{S}_r\Lambda^0$ is $r+1$ on affinely mapped elements, but only $\max(2, \lfloor r/n \rfloor + 1)$ on multilinearly mapped elements.
- The rate for $\mathcal{Q}_r^-\Lambda^k$, $k > 0$, is r on affinely mapped elements, $r-k+1$ on multilinearly mapped elements.
- The rate for $\mathcal{P}_r\Lambda^n = \mathcal{S}_r\Lambda^n$ is $r+1$ for affinely mapped elements, $\lfloor r/n \rfloor - n + 2$ for multilinearly mapped.



DNA-Boffi-Bonizzoni 2012


$\mathcal{P}_r^-\Lambda^k$
$\mathcal{P}_r\Lambda^k$
$\mathcal{Q}_r^-\Lambda^k$
$\mathcal{S}_r\Lambda^k$
$k = 0$ $k = 1$ $k = 2$ $k = 3$
$n = 1$ $n = 2$ $n = 3$
$r = 1$ $r = 2$ $r = 3$

Future directions

- Hermite finite elements (Argyris)
- Smooth spline spaces (isogeometric elements)
- Nonconforming finite elements (Crouzeix–Raviart, Morley)
- Other complexes, such as the *Stokes complex* (J. Evans '11)

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H^1(\text{curl}, \Omega) \xrightarrow{\textit{curl}} H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \to 0$$

$$H^1(\text{curl}, \Omega) = \{\, u \in H(\text{curl}, \Omega) \mid \text{curl}\, u \in H^1(\Omega; \mathbb{R}^3) \,\}$$

or the *elasticity complex*

$$0 \to H^1(\Omega; \mathbb{R}^3) \xrightarrow{\epsilon} H(J, \Omega; \mathbb{S}^{3\times3}) \xrightarrow{J} H(\text{div}, \Omega; \mathbb{S}^{3\times3}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \to 0$$

$J = \text{curl}\, T\, \text{curl} =$ St. Venant tensor $=$ linearized Einstein tensor