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PREPRINT 2000-09

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Göteborg, January 2001

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NO 2000-09
ISSN 1404–4382

Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
`www.phi.chalmers.se`

Printed in Sweden
Chalmers University of Technology
Göteborg, Sweden 2001

DISCONTINUOUS GALERKIN AND THE CROUZEIX-RAVIART ELEMENT: APPLICATION TO ELASTICITY

PETER HANSBO AND MATS G. LARSON

Abstract. In this work we propose a discontinuous Galerkin method for linear elasticity, based on discontinuous piecewise linear approximation of the displacements. The method is based on the classical Nitsche method with a term stabilizing (or penalizing) discontinuities of a special form. We show optimal order a priori error estimates, uniform in the incompressible limit, and thus locking is avoided. The discontinuous Galerkin method is closely related to the non-conforming Crouzeix-Raviart (CR) element, which in fact is obtained when one of the stabilizing parameters tends to infinity. In the case of the elasticity operator, for which the CR element is not stable in that it does not fulfill a discrete Korn's inequality, the discontinuous framework naturally suggests the appearance of (weakly consistent) stabilization terms. Thus, a stabilized version of the CR element, which does not lock, can be used for both compressible and (nearly) incompressible elasticity. Numerical results supporting these assertions are included. The analysis directly extends to higher order elements and three spatial dimensions.

1. Introduction

In a discontinuous Galerkin method the approximation space typically consists of discontinuous piecewise polynomials with boundary conditions and continuity on inter-element boundaries weakly enforced through the bilinear form. For second order problems these methods appear to origin from the work of Nitsche [9], where a consistent method for weak imposement of Dirichlet conditions was introduced. Later, similar techniques to enforce continuity on inter element boundaries were introduced and analyzed, see for instance Wheeler [13] and Arnold [1]. Recently there has been a growing interest in discontinuous Galerkin methods for a variety of different applications, see the conference proceedings [14] for an overview of recent work.

In this paper we propose and analyze a discontinuous Galerkin method for linear elasticity, which has optimal order and does not lock in the incompressible limit. The method is related to the earlier work of Hansbo and Larson [7], but uses a different stabilization (or

Date: January 29, 2001.

Key words and phrases. Crouzeix-Raviart element, Nitsche's method, discontinuous Galerkin, incompressible elasticity.

Peter Hansbo, Department of Solid Mechanics, Chalmers University of Technology, S–412 96 Göteborg, Sweden, *email*:hansbo@solid.chalmers.se

Mats G. Larson, Department of Mathematics, Chalmers University of Technology, S–412 96 Göteborg, Sweden, *email*:mgl@math.chalmers.se

Supported by the Swedish Foundation for Strategic Research and the Swedish Council for Engineering Research.

penalization) of discontinuities. There is a natural connection between the discontinuous Galerkin method proposed here and the classical nonconforming Crouzeix-Raviart (CR) element, see [4]. The lowest order CR element is a simple nonconforming finite element for triangular elements with nodes situated at the midpoints of the element sides, which can be used for the Poisson problem and the Stokes or Navier-Stokes problems [4, 11]. Further, Brenner and Sung [3] used the CR element to construct a locking free method for the pure displacement problem of almost incompressible elasticity. However, for the traction problem in elasticity the CR element is known to be unstable, since it can not control the rigid body rotations, cf. Hughes [8, Sec. 4.7]. In the two dimensional case, Falk [5] obtained a stable version of the CR element by splitting the elasticity operator and projecting one part of the operator onto a macro element. In this paper, we instead obtain a stabilized method for the CR element, by simply using the CR element in the discontinuous Galerkin method. In fact, the CR element approximation is obtained as a certain stabilization parameter tends to infinity in the discontinuous Galerkin method. Thus, error estimates for the nonconforming method based on the CR element is obtained as a special case of the error estimates for the discontinuous Galerkin method. Although we present the analysis of linear elements in two spatial dimensions it directly extends to higher order elements and three spatial dimensions.

The paper is organized as follows: in Section 2 we introduce the equations of elasticity and the discontinuous Galerkin method, and we prove a priori error estimates; in Section 3 we introduce the stabilized method for the nonconforming CR element; and in Section 4 we present numerical examples illustrating our results.

2. A discontinuous Galerkin method for elasticity

2.1. **The equations of elasticity.** We consider the equations of linear elasticity describing the deplacement of an elastic body occupying a domain Ω in two spatial dimensions: find the displacement $\boldsymbol{u} = [u_i]_{i=1}^2$ and the symmetric stress tensor $\boldsymbol{\sigma} = [\sigma_{ij}]_{i,j=1}^2$ such that

$$\boldsymbol{\sigma} = \lambda\ \nabla\cdot\boldsymbol{u}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.1}$$

$$-\nabla\cdot\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega_{\mathrm{D}}, \tag{2.3}$$

$$\boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{h} \quad \text{on } \partial\Omega_{\mathrm{N}}. \tag{2.4}$$

Here $\boldsymbol{\varepsilon}(\boldsymbol{u}) = [\varepsilon_{ij}(\boldsymbol{u})]_{i,j=1}^2$ is the strain tensor with components

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

$\nabla\cdot\boldsymbol{\sigma} = \left[\sum_{j=1}^2 \partial\sigma_{ij}/\partial x_j\right]_{i=1}^2$, $\boldsymbol{I} = [\delta_{ij}]_{i,j=1}^2$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, $\boldsymbol{f}$ and $\boldsymbol{h}$ are given loads, and $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$. Furthermore, λ and μ are the Lamé constants, satisfying $0 < \mu_1 < \mu < \mu_2$ and $0 < \lambda < \infty$. In terms of the modulus of elasticity, E, and Poisson's ratio, ν, we have, in the case of plane strain, that

$\lambda = E\nu/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$. Incompressible behavior is obtained as the parameter $\lambda \to \infty$, i.e., as $\nu \to 1/2$.

2.2. **Formulation of the discontinuous Galerkin method.** Consider a subdivision $\mathcal{T} = \{T\}$ of Ω of Ω into a geometrically conforming finite element mesh. with h_T the diameter of triangle T and global mesh size parameter $h = \max_{T\in\mathcal{T}} h_T$. For simplicity, we assume that $\sqcup$ is quasiuniform. Let

$$\boldsymbol{DF} = \{\boldsymbol{v} \in [L^2(\Omega)]^2 : \ \boldsymbol{v}|_T \in P^1(T) \text{ for all } T \in \mathcal{T}\},$$

be the space of piecewise linear discontinuous functions. The set of edges in the mesh is denoted by $\mathcal{E} = \{E\}$ and we split $\mathcal{E}$ into three disjoint subsets

$$\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N,$$

where $\mathcal{E}_I$ is the set of edges in the interior of Ω, $\mathcal{E}_D$ is the set of edges on the Dirichlet part of the boundary $\partial\Omega_D$, and $\mathcal{E}_N$ is the set of edges in the Neumann part of the boundary $\partial\Omega_N$. Further, with each edge we associate a fixed unit normal $\boldsymbol{n}$ such that for edges on the boundary $\boldsymbol{n}$ is the exterior unit normal. We denote the jump of a function $\boldsymbol{v} \in \boldsymbol{DF}$ at an edge E by $[\boldsymbol{v}] = \boldsymbol{v}^+ - \boldsymbol{v}^-$ for $E \in \mathcal{E}_I$ and $[\boldsymbol{v}] = \boldsymbol{v}^+$ for $E \in \mathcal{E}_D$, and the average $\langle \boldsymbol{v} \rangle = (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2$ for $E \in \mathcal{E}_I$ and $\langle \boldsymbol{v} \rangle = \boldsymbol{v}^+$ for $E \in \mathcal{E}_D$, where $\boldsymbol{v}^\pm = \lim_{\epsilon\downarrow 0} \boldsymbol{v}(\boldsymbol{x} \mp \epsilon\, \boldsymbol{n})$ with $\boldsymbol{x} \in E$.

The discontinuous Galerkin method reads: find $\boldsymbol{U} \in \boldsymbol{DF}$ such that

$$a(\boldsymbol{U}, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{DF}. \tag{2.5}$$

The bilinear form is defined by

$$\begin{aligned} a(\boldsymbol{U}, \boldsymbol{v}) = & \sum_{T\in\mathcal{T}} (\boldsymbol{\sigma}(\boldsymbol{U}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T \\ & - \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (\langle \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{U})\rangle, [\boldsymbol{v}])_E + (\langle \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{v})\rangle, [\boldsymbol{U}])_E \\ & + (2\mu+\lambda)\gamma_0 \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (h^{-1}[P_0\boldsymbol{U}], [P_0\boldsymbol{v}])_E \\ & + 2\mu\gamma_1 \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (h^{-1}[\boldsymbol{U}], [\boldsymbol{v}])_E, \end{aligned} \tag{2.6}$$

and the linear functional is defined by

$$l(\boldsymbol{v}) = \sum_{T\in\mathcal{T}} (\boldsymbol{f}, \boldsymbol{v})_T + \sum_{E\in\mathcal{E}_N} (\boldsymbol{h}, \boldsymbol{v})_E. \tag{2.7}$$

Here $(\boldsymbol{v}, \boldsymbol{w})_T = \int_T \sum_{ij} \boldsymbol{v}_{ij}\boldsymbol{w}_{ij}$, for 2-tensors $\boldsymbol{v}, \boldsymbol{w}$; $(\boldsymbol{v}, \boldsymbol{w})_E = \int_E \sum_i \boldsymbol{v}_i\boldsymbol{w}_i$, for vectors $\boldsymbol{v}, \boldsymbol{w}$; P_0 is the L^2-projection onto constants on each edge E, i.e.,

$$P_0\boldsymbol{v}|_E = \frac{1}{|E|}\int_E \boldsymbol{v}, \tag{2.8}$$

with $|E|$ the length of E; h is defined by

$$h|_E = \left(|T^+| + |T^-|\right)/(2\,|E|) \quad \text{for } E = \partial T^+ \cap \partial T^-, \tag{2.9}$$

with $|T|$ the area of T, on each edge.

Using Green's formula, we readily establish the following proposition.

Proposition 2.1. *The method* (2.5) *is consistent in the sense that*

$$a(\boldsymbol{u} - \boldsymbol{U}, \boldsymbol{v}) = 0 \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{DF},$$

and $\boldsymbol{u}$ *sufficiently regular.*

2.3. **A priori error estimates.** For the purpose of error analysis, we introduce the following mesh dependent energy norm

$$|||\boldsymbol{v}|||^2 = \sum_{T\in\mathcal{T}} (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T + \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (h^{-1}\,[\boldsymbol{v}]\,, [\boldsymbol{v}])_E, \tag{2.10}$$

and the edge norm

$$\|\boldsymbol{v}\|^2 = \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} \|\boldsymbol{v}\|^2_{L^2(E)}. \tag{2.11}$$

The mesh dependent norm $|||\cdot|||$ can be used to bound the broken $H^1(\Omega)$ norm on $\boldsymbol{DF}$, which we show in the following proposition.

Proposition 2.2. *There is a constant* c*, independent of* h*,* μ*, and* λ *such that*

$$\sum_{T\in\mathcal{T}} \|\boldsymbol{v}\|^2_{H^1(T)} \le c|||\boldsymbol{v}|||^2 \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{DF}. \tag{2.12}$$

Proof. Assume that the right-hand side of (2.12) is zero. Note that $(\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T = 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T + \lambda(\nabla\cdot\boldsymbol{v}, \nabla\cdot\boldsymbol{v})_T$ and, since $0 < \mu_1 < \mu$ for some positive constant μ_1, $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(T)} = 0$, and thus $\boldsymbol{v}|_T \in \boldsymbol{RM}(T)$, where

$$\boldsymbol{RM}(T) = \{\boldsymbol{v} \in P^1(T) :\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a}_T + b_T\,(-x_2, x_1),\ \boldsymbol{a}_T \in \mathbb{R}^2,\ b_T \in \mathbb{R}\}, \tag{2.13}$$

is the space of linearized rigid body motions on T. Next, using $\|[\boldsymbol{v}]\|_{L^2(E)} = 0$, for all $E \in \mathcal{E}_I$, we conclude that there are constants $\boldsymbol{a}$ and b such that $\boldsymbol{a} = \boldsymbol{a}_T$ and $b = b_T$, for all triangles T. Furthermore, from $\|\boldsymbol{v}\|_{L^2(E)} = 0$ for $E \in \mathcal{E}_D$, it follows that $\boldsymbol{a} = \boldsymbol{0}$ and $b = 0$. Thus, if the right-hand side of (2.12) is zero, so is the left-hand side. Finally, finite dimensionality, together with scaling yields the result. □

In order to show that the method (2.5) is stable, we shall show that $a\,(\cdot,\cdot)$ is coercive with respect to the norm $|||\cdot|||$, given that γ_0 is sufficiently large and γ_1 is positive and bounded away from zero. We also give a bound on the piecewise constant part of the jumps at each edge.

Proposition 2.3. *If* $\gamma_0 > c_0$*, with* c_0 *sufficiently large and* $\gamma_1 \ge c_1 > 0$*, then the following estimates hold*

$$c\,|||\boldsymbol{v}|||^2 + \alpha(\lambda,\gamma_0)\|h^{-1/2}[P_0\boldsymbol{v}]\|^2_{\mathcal{E}_I\cup\mathcal{E}_D} \le a(\boldsymbol{v},\boldsymbol{v}), \tag{2.14}$$

for all $\boldsymbol{v} \in \boldsymbol{DF}$. *Here* $\alpha(\lambda, \gamma_0) = (2\mu + \lambda)(\gamma_0 - c_0)$ *and the constant* c *is independent of* $h, \gamma_0, \gamma_1, \mu,$ *and* λ.

Proof. We first note that the following inverse estimate holds

$$\|h^{1/2}\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\rangle\|^2_{\mathcal{E}_I \cup \mathcal{E}_D} \leq c \sum_{T \in \mathcal{T}} \|\boldsymbol{\sigma}(\boldsymbol{v})\|^2_T. \tag{2.15}$$

This inequality is proved by scaling and finite dimensionality. Next we note that $(2\mu + \lambda)^{-1}\|\boldsymbol{\sigma}(\boldsymbol{v})\|^2_T \leq (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T$, and thus we conclude that

$$\frac{1}{2\mu + \lambda}\|h^{1/2}\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\rangle\|^2_{\mathcal{E}_I \cup \mathcal{E}_D} \leq c \sum_{T \in \mathcal{T}} (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T. \tag{2.16}$$

Next, using that $\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\rangle$ is constant we have, for each $E \in \mathcal{E}_I \cup \mathcal{E}_D$ and for $\delta > 0$, that

$$\begin{aligned} 2(\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\rangle, [\boldsymbol{v}])_E &= 2(\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\rangle, [P_0\boldsymbol{v}])_E \\ &\leq \delta(2\mu + \lambda)\|h^{1/2}\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\rangle\|^2_E \\ &\quad + \delta^{-1}(2\mu + \lambda)^{-1}\|h^{-1/2}[P_0\boldsymbol{v}]\|^2_E, \end{aligned} \tag{2.17}$$

where we used the Cauchy-Schwarz inequality followed by the arithmetic-geometric mean inequality. Using these estimates we obtain

$$\begin{aligned} a(\boldsymbol{v}, \boldsymbol{v}) &\geq (1 - c\delta) \sum_{T \in \mathcal{T}} (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_T \\ &\quad + (2\mu + \lambda)(\gamma_0 - \delta^{-1})\|h^{-1/2}[P_0\boldsymbol{v}]\|^2_{\mathcal{E}_I \cup \mathcal{E}_D} \\ &\quad + 2\mu\gamma_1\|h^{-1/2}[\boldsymbol{v}]\|^2_{\mathcal{E}_I \cup \mathcal{E}_D} \\ &\geq c|||\boldsymbol{v}|||^2 + (2\mu + \lambda)(\gamma_0 - \delta^{-1})\|h^{-1/2}[P_0\boldsymbol{v}]\|^2_{\mathcal{E}_I \cup \mathcal{E}_D}. \end{aligned}$$

Choosing δ small enough we obtain the desired estimates. □

For the proof of our a priori error estimate we introduce the interpolation operator $\boldsymbol{\pi} : [H^2(\Omega)]^2 \to \boldsymbol{DF}$ introduced by Crouzeix and Raviart in [4], and constructed as follows. At the midpoint $\boldsymbol{x}_m$ of the edge E, let

$$\boldsymbol{\pi u}(\boldsymbol{x}_m) := \frac{1}{|E|} \int_E \boldsymbol{u},$$

from which it follows, by application of Gauss' theorem, cf. [4], that

$$\nabla \cdot (\boldsymbol{\pi u})|_T = \frac{1}{|T|} \int_T \nabla \cdot \boldsymbol{u}. \tag{2.18}$$

For this interpolant, we have the following basic estimates.

Lemma 2.1. *The following estimates hold*

$$\|\boldsymbol{u} - \boldsymbol{\pi u}\|_{L_2(T)} + h_T\|\boldsymbol{u} - \boldsymbol{\pi u}\|_{H^1(T)} \leq Ch_T^2\|\boldsymbol{u}\|_{H^2(T)}, \tag{2.19}$$

$$\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{\pi u})\|_{L_2(T)} + h_T\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{\pi u})\|_{H^1(T)} \leq Ch_T\|\nabla \cdot \boldsymbol{u}\|_{H^1(T)}. \tag{2.20}$$

Proof. The proof of (2.19) can be found in [4], and (2.20) follows from the fact that $\nabla \cdot (\boldsymbol{\pi u})|_T$ is the L_2-projection of $\nabla \cdot \boldsymbol{u}$ onto the space of constants on T, cf. (2.18). □

For our a priori error estimate we shall need the following lemma.

Lemma 2.2. *The following estimates hold*

$$|||\boldsymbol{u} - \boldsymbol{\pi u}||| \leq Ch\Big((2\mu)^{1/2}\|\boldsymbol{u}\|_{H^2(\Omega)} + \lambda^{1/2}\|\nabla \cdot \boldsymbol{u}\|_{H^1(\Omega)}\Big), \tag{2.21}$$

$$\|\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{\pi u})\rangle\|_{\mathcal{E}_I \cup \mathcal{E}_D} \leq Ch\Big(2\mu\|\boldsymbol{u}\|_{H^2} + \lambda\|\nabla \cdot \boldsymbol{u}\|_{H^1(\Omega)}\Big). \tag{2.22}$$

Proof. We first recall the trace inequality

$$\|\boldsymbol{w}\|^2_{L^2(\partial T)} \leq C\|\boldsymbol{w}\|_{L^2(T)}\left(\|\boldsymbol{w}\|_{H^1(T)} + h_T^{-1}\|\boldsymbol{w}\|_{L^2(T)}\right) \quad \forall \boldsymbol{w} \in H^1(T).$$

For the first inequality (2.21), the interior part is estimated directly using Lemma 2.1. For the contribution from the edges, we have using the triangle inequality

$$\|h^{-1/2}[\boldsymbol{u} - \boldsymbol{\pi u}]\|_{\mathcal{E}_I \cup \mathcal{E}_D} \leq \sum_{T \in \mathcal{T}} \|h^{-1/2}(\boldsymbol{u} - \boldsymbol{\pi u})\|_{\partial T}.$$

Next using the trace inequality we get

$$\|h^{-1/2}(\boldsymbol{u} - \boldsymbol{\pi u})\|^2_{\partial T} \leq h_T^{-1}\|\boldsymbol{u} - \boldsymbol{\pi u}\|_{L^2(T)}\left(\|\boldsymbol{u} - \boldsymbol{\pi u}\|_{H^1(T)} + h_T^{-1}\|\boldsymbol{u} - \boldsymbol{\pi u}\|_{L^2(T)}\right),$$

and finally invoking Lemma 2.1 we get the desired estimate. The second inequality (2.22), follows in a similar fashion. □

We are now ready to show our main result.

Theorem 2.1. *With* $\boldsymbol{u}$ *the solution of* (2.1) *and* $\boldsymbol{U}$ *the solution of* (2.5), *we have under the assumptions of Proposition 2.3,*

$$|||\boldsymbol{u} - \boldsymbol{U}||| + \alpha(\lambda, \gamma_0)^{1/2}\|h^{-1/2}[P_0\boldsymbol{U}]\|_{\mathcal{E}_I \cup \mathcal{E}_D} \leq ch\Big((2\mu)^{1/2}\|\boldsymbol{u}\|_{H^2} + \lambda\|\nabla \cdot \boldsymbol{u}\|_{H^1}\Big),$$

where $\alpha(\cdot, \cdot)$ *is defined in Proposition 2.3 and* c *is a constant independent of* h, γ_0, γ_1, μ, *and* λ.

Proof. By the triangle inequality, we have that

$$|||\boldsymbol{u} - \boldsymbol{U}||| \leq |||\boldsymbol{U} - \boldsymbol{\pi u}||| + |||\boldsymbol{u} - \boldsymbol{\pi u}|||, \tag{2.23}$$

and from coercivity, Proposition 2.3, and consistency, Proposition 2.1, it follows that

$$\begin{aligned} c|||\boldsymbol{U} - \boldsymbol{\pi u}|||^2 &+ \alpha(\lambda, \gamma_0)\|h^{-1/2}[P_0\boldsymbol{U}]\|^2_{\mathcal{E}_I \cup \mathcal{E}_D} \\ &\leq a(\boldsymbol{U} - \boldsymbol{\pi u}, \boldsymbol{U} - \boldsymbol{\pi u}) = a(\boldsymbol{u} - \boldsymbol{\pi u}, \boldsymbol{U} - \boldsymbol{\pi u}). \end{aligned} \tag{2.24}$$

To estimate the right hand side $a(\boldsymbol{u} - \boldsymbol{\pi u}, \boldsymbol{U} - \boldsymbol{\pi u})$, we first note that

$$(\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{U} - \boldsymbol{\pi u})\rangle, [\boldsymbol{u} - \boldsymbol{\pi u}])_E = 0, \tag{2.25}$$

$$(h^{-1}[P_0(\boldsymbol{U} - \boldsymbol{\pi u})], [P_0(\boldsymbol{u} - \boldsymbol{\pi u})])_E = 0, \tag{2.26}$$

and then use the Cauchy-Schwarz inequality to get

$$a(\boldsymbol{u}-\boldsymbol{\pi u},\boldsymbol{U}-\boldsymbol{\pi u}) \le |||\boldsymbol{U}-\boldsymbol{\pi u}|||\Big(|||\boldsymbol{u}-\boldsymbol{\pi u}||| + \|h^{1/2}\langle \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{\pi u})\rangle\|_{\mathcal{E}_I\cup\mathcal{E}_D}\Big). \tag{2.27}$$

Combining (2.24) and (2.27), we find that

$$|||\boldsymbol{U}-\boldsymbol{\pi u}||| \le c\Big(|||\boldsymbol{u}-\boldsymbol{\pi u}||| + \|h^{1/2}\langle \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{\pi u})\rangle\|_{\mathcal{E}_I\cup\mathcal{E}_D}\Big). \tag{2.28}$$

and thus

$$a(\boldsymbol{u}-\boldsymbol{\pi u},\boldsymbol{U}-\boldsymbol{\pi u}) \le c\Big(|||\boldsymbol{u}-\boldsymbol{\pi u}||| + \|h^{1/2}\langle \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{\pi u})\rangle\|_{\mathcal{E}_I\cup\mathcal{E}_D}\Big)^2.$$

Finally, using the interpolation estimates of Lemma 2.2, the bound follows immediately. □

Combining the error estimate in Theorem 2.1 with the elliptic regularity estimate

$$\|\boldsymbol{u}\|_{H^2(\Omega)} + \lambda\|\nabla\cdot\boldsymbol{u}\|_{H^1(\Omega)} \le c\Big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|\boldsymbol{h}\|_{H^{1/2}(\partial\Omega_N)}\Big), \tag{2.29}$$

cf. Vogelius [12], we obtain the following estimate in terms of data, which shows that the method does not lock as $\lambda\to\infty$.

Corollary 2.1. *There is a constant c, independent of h, γ_0, γ_1, μ, and λ such that*

$$|||\boldsymbol{u}-\boldsymbol{U}||| + \alpha(\lambda,\gamma_0)^{1/2}\|h^{-1/2}[P_0\boldsymbol{U}]\|_{\mathcal{E}_I\cup\mathcal{E}_D} \le ch\Big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|\boldsymbol{h}\|_{H^{1/2}(\partial\Omega_N)}\Big). \tag{2.30}$$

3. A nonconforming Galerkin method based on the Crouzeix-Raviart element

Restricting the discontinuous Galerkin method to the space of Crouzeix-Raviart functions

$$\boldsymbol{CR} = \{\boldsymbol{v}\in\boldsymbol{DF} : [P_0\boldsymbol{v}] = 0, \text{ for all } E\in\mathcal{E}_I\cup\mathcal{E}_D\}, \tag{3.1}$$

we obtain the following simplified scheme. Find $\boldsymbol{U}\in\boldsymbol{CR}$ such that

$$a(\boldsymbol{U},\boldsymbol{v}) = l(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v}\in\boldsymbol{CR}, \tag{3.2}$$

where

$$a(\boldsymbol{U},\boldsymbol{v}) = \sum_{T\in\mathcal{T}}(\boldsymbol{\sigma}(\boldsymbol{U}),\boldsymbol{\varepsilon}(\boldsymbol{v})) + 2\mu\gamma_1\sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D}(h^{-1}[\boldsymbol{U}]\cdot[\boldsymbol{v}])_E, \tag{3.3}$$

and

$$l(\boldsymbol{v}) = \sum_{T\in\mathcal{T}}(\boldsymbol{f},\boldsymbol{v})_T + \sum_{E\in\mathcal{E}_N}(\boldsymbol{h},\boldsymbol{v})_E. \tag{3.4}$$

Using the theory developed in the previous section to this method we obtain the following theorem.

Theorem 3.1. *The discontinuous Galerkin method with the CR-element* (3.2) *is the limit of the discontinuous Galerkin method* (2.5) *as* $\gamma_0 \to \infty$. *If* $\gamma_1 \geq c_1 > 0$, *then* (3.2) *has a unique solution* $\boldsymbol{U}$ *and the following error estimate holds*

$$|||\boldsymbol{u} - \boldsymbol{U}||| \leq ch\Big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|\boldsymbol{h}\|_{H^{1/2}(\partial\Omega_N)}\Big), \tag{3.5}$$

with c *independent of* μ, λ, *and* h.

Proof. The fact that the solution to the discontinuous Galerkin method (2.5) tends to (3.2) and the error estimate follow from Theorem 2.1, since the constant is independent of γ_0 we just let γ_0 tend to infinity. □

Remark. Our analysis extends directly to the case of higher order polynomials. For odd order of polynomials we get the CR-family of elements, while for even the situation is not so simple, due to the fact that the values of a polynomial at the Gauss points on the edges of a triangle is dependent in that case. For polynomials of order two we refer to Fortin and Soulie [6], where it is shown that one obtains the usual continuous quadratic polynomials together with a nonconforming bubble on each element.

Remark. On rectangular elements we instead obtain the element proposed by Rannacher and Turek in [10].

4. Examples

4.1. **Convergence in a smooth case.** We consider the unit square $(0,1) \times (0,1)$ with $\boldsymbol{u} = 0$ on the boundary and with

$$\boldsymbol{f} = ((\lambda + \mu)(1 - 2\,x_1)(1 - 2\,x_2), -2\mu\, x_2\,(1 - x_2) - 2\,(\lambda + 2\mu)\, x_1\,(1 - x_1))$$

corresponding to the exact solution

$$\boldsymbol{u} = (0, -x_1\, x_2\,(1 - x_1)\,(1 - x_2))\,.$$

The material data were chosen as $E = 1000$, and $\nu = 1/10$, where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{2\,(1+\nu)}.$$

A mesh with the exact solution plotted in the nodes is given in Figure 1. In Figure 2 we show the stability problem of the CR element corresponding to choosing γ_1 too small, and the stabilizing effect of the boundary jumps with an adequate choice of γ_1. In Figure 3 we show the convergence of the CR method (with $\gamma_1 = 0.5$), and second order convergence is attained, as expected. We also give the convergence rate of the corresponding conforming linear finite element method. It is noticeable that the error relative to the number of unknowns is almost identical for the nonconforming and conforming method.

4.2. **Locking.** On the unit square $(0,1)\times(0,1)$, we prescribe the boundary data as follows. At $x_1 = 0, x_1 = 1, x_2 = 0$ we set $\boldsymbol{u} = 0$, while at $x_2 = 1$, for $0 < x_1 < 1$,we set $\boldsymbol{u} = (1,0)$ to obtain the closed cavity flow problem. We compare the linear conforming finite element method with the stabilized CR approximation ($\gamma_1 = 0.5$) for $\nu \to 1/2$. The CR element does not lock, while the conforming method, for $\nu = 0.4999$, shows clear signs of locking.

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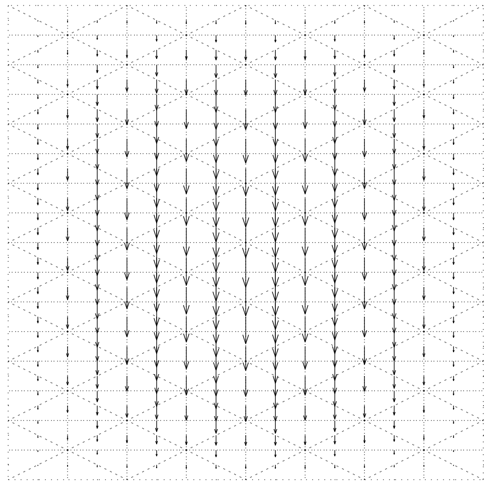


FIGURE 1. Exact solution.

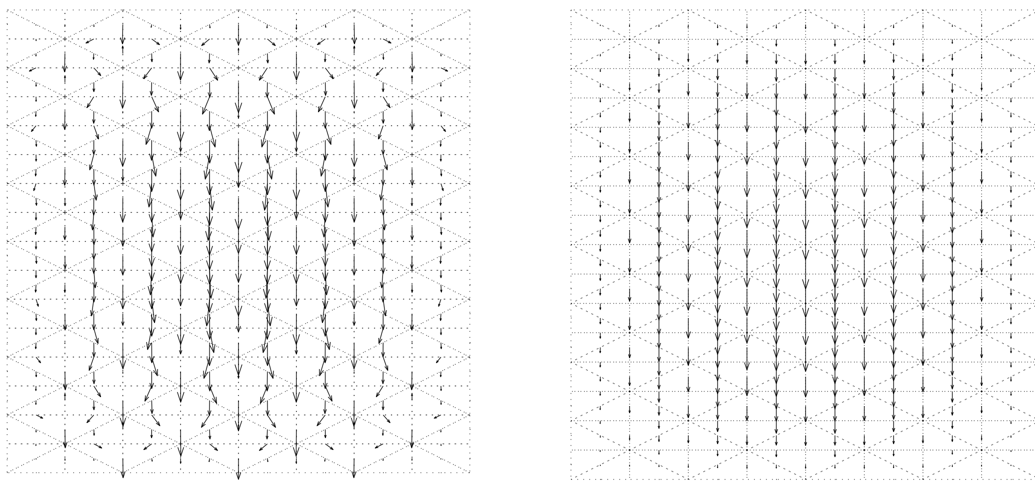


FIGURE 2. Unstable solution (a) corresponding to choosing $\gamma_1 << \mu$, and (b) stable solution for $\gamma_1 = \mu$.

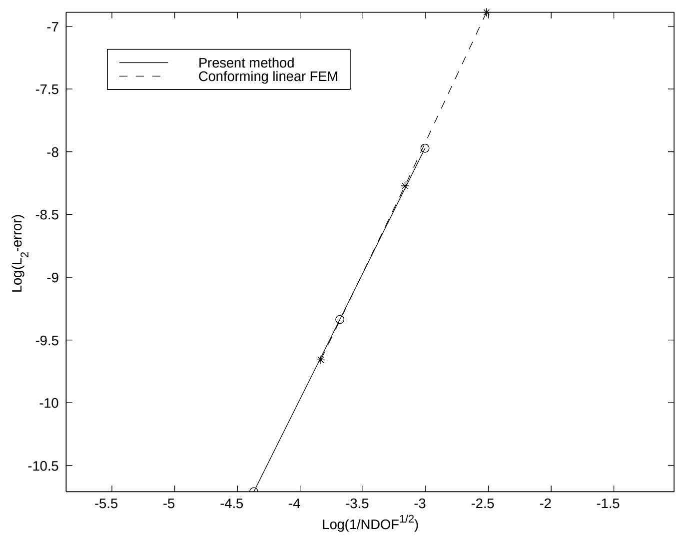
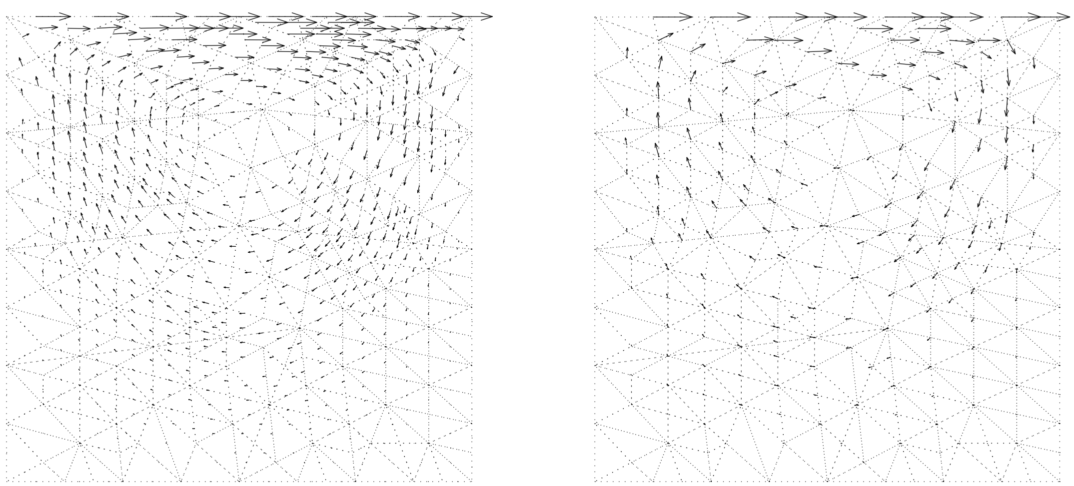
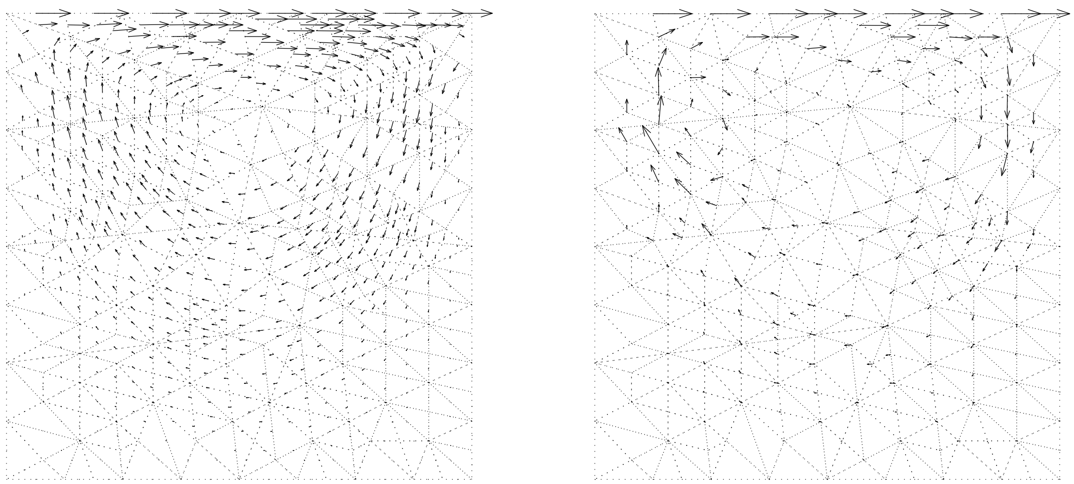


FIGURE 3. Second order convergence in the $L_2(\Omega)$−norm. The error relative to the number of degrees of freedom is comparable to a conforming finite element method.



(a) Nonconforming (b) Conforming

FIGURE 4. Nonconforming (a) and conforming (b) solutions for $\nu = 0.49$.



(a) Nonconforming (b) Conforming

FIGURE 5. Nonconforming (a) and (locked) conforming (b) solutions for $\nu = 0.4999$.

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