0 5 setgray0 0 5 setgray1

A compiler for variational forms - practical results

USNCCM8

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Overview

- Part I - **FEniCS** Form Compiler (FFC):
  - Motivation for FFC (generality of FEM)
  - Introduction to FFC
  - Benchmarks (FFC vs. quadrature)
- Part II - Application (Elasto-Plasticity):
  - Motivation for elasto-plastic model
  - Implementation of elasto-plastic model in FFC
  - Benchmarks (FFC vs. mass-spring)
  - Future work

Motivation for FFC

FEniCS project: Automation of Computational Mathematical Modeling (ACMM)

Finite Element Method: General method for automating discretization of differential equations

This generality is seldom reflected in software

Reasons: conceptual complexity, hand-written routines often outperform general routines

How can we overcome these difficulties?

Through a Form Compiler which automatically generates an optimal Finite Element routine (assembly)

Motivation for FFC

Advantages of compilation:

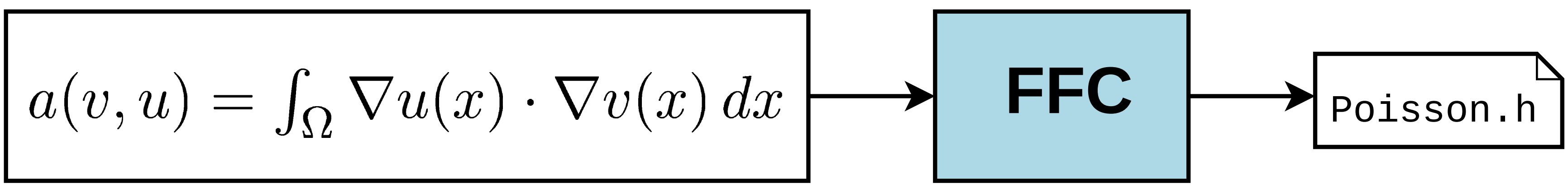
- A form compiler can be written in a high-level language and/or with high-level data structures which eases conceptual abstraction.
- A form compiler can pre-compute quantities which are known at compile time.

Disadvantages of compilation:

- Forms cannot easily be modified during run time.

FFC: the FEniCS Form Compiler

- Automates a key step in the implementation of finite element methods for partial differential equations
- Input: a variational form and a finite element
- Output: optimal C/C++




```
>> ffc [-l language] poisson.form
```

Basic example: Poisson's equation

- Strong form: Find $u \in \mathcal{C}^2(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$ such that

$$-\Delta u = f \quad \text{in } \Omega$$

- Weak form: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx = \int_\Omega f(x) v(x) \, dx \quad \text{for all } v \in H_0^1(\Omega)$$

- Standard notation: Find $u \in V$ such that

$$a(v, u) = L(v) \quad \text{for all } v \in \hat{V}$$

with $a : \hat{V} \times V \to \mathbb{R}$ a *bilinear form* and $L : \hat{V} \to \mathbb{R}$ a *linear form* (functional)

Obtaining the discrete system

Let V and $\hat{V}$ be discrete function spaces. Then

$$a(v, U) = L(v) \quad \text{for all } v \in \hat{V}$$

is a discrete linear system for the approximate solution $U \approx u$.

With $V = \mathrm{span}\{\phi_i\}_{i=1}^M$ and $\hat{V} = \mathrm{span}\{\hat{\phi}_i\}_{i=1}^M$, we obtain the linear system

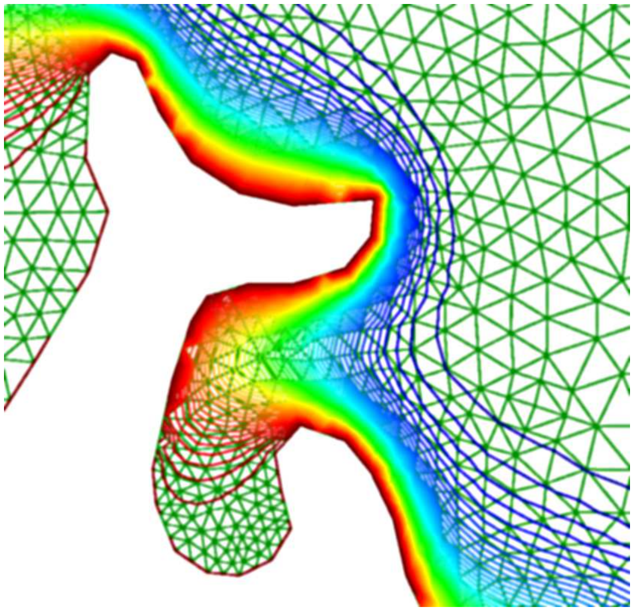
$$Ax = b$$

for the degrees of freedom $x = (x_i)$ of $U = \sum_{i=1}^M x_i \phi_i$, where

$$A_{ij} = a(\hat{\phi}_i, \phi_j)$$

$$b_i = L(\hat{\phi}_i)$$

Computing the linear system: assembly



Noting that $a(v,u) = \sum_{K \in \mathcal{T}} a_K(v,u)$, the matrix A can be assembled by

$A = 0$
`for all elements` $K \in \mathcal{T}$
  A += A^K

The *element matrix* A^K is defined by

$$A_{ij}^K = a_K(\hat{\phi}_i, \phi_j)$$

for all local basis functions $\hat{\phi}_i$ and ϕ_j on K

Multi-linear forms

Consider a multi-linear form

$$a : V_1 \times V_2 \times \cdots \times V_r \to \mathbb{R}$$

with $V_1, V_2, \ldots, V_r$ function spaces on the domain Ω

- Typically, $r = 1$ (linear form) or $r = 2$ (bilinear form)
- Assume $V_1 = V_2 = \cdots = V_r = V$ for ease of notation

Want to compute the rank r *element tensor* A^K defined by

$$A_i^K = a_K(\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_r})$$

with $\{\phi_i\}_{i=1}^n$ the local basis on K and multi-index $i = (i_1, i_2, \ldots, i_r)$

Tensor representation

In general, the element tensor A^K can be represented as the product of a *reference tensor* A^0 and a *geometry tensor* G_K:

$$A_i^K = A_{i\alpha}^0 G_K^\alpha$$

- A^0: a tensor of rank $|i| + |\alpha| = r + |\alpha|$
- G_K: a tensor of rank $|\alpha|$

Basic idea:

- Precompute A^0 at compile-time
- Generate optimal code for run-time evaluation of G_K and the product $A_{i\alpha}^0 G_K^\alpha$

Example: Poisson

- Form:

$$a(v, u) = \int_\Omega \nabla v(x) \cdot \nabla u(x) \, dx$$

- Evaluation:

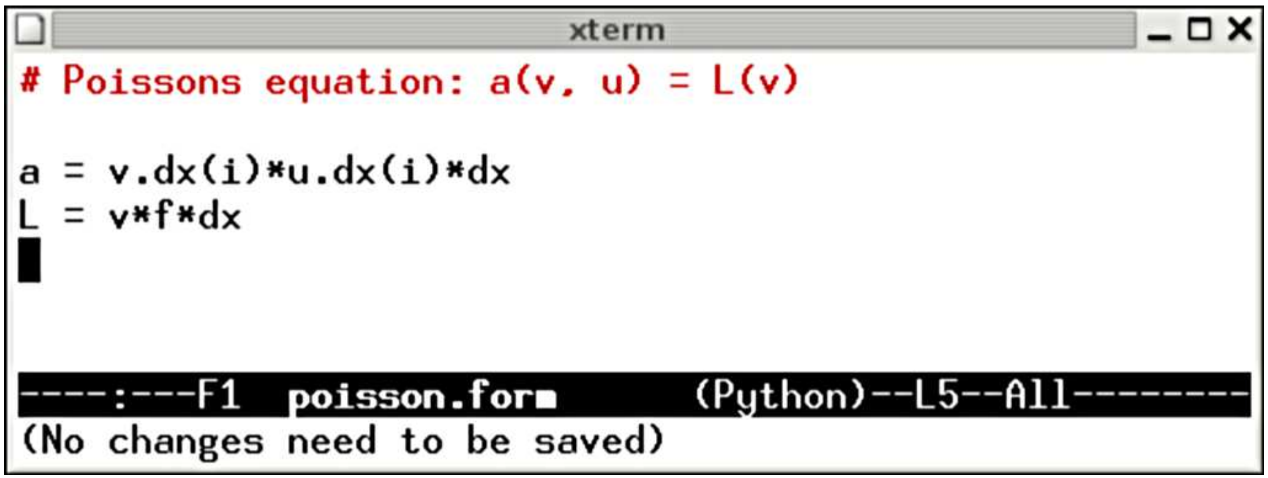
$$A_i^K = \int_K \nabla \phi_{i_1}(x) \cdot \nabla \phi_{i_2}(x) \, dx$$

$$= \det F_K' \frac{\partial X_{\alpha_1}}{\partial x_\beta} \frac{\partial X_{\alpha_2}}{\partial x_\beta} \int_{K_0} \frac{\partial \Phi_{i_1}}{\partial X_{\alpha_1}} \frac{\partial \Phi_{i_2}}{\partial X_{\alpha_2}} \, dX = A^0_{i\alpha} G_K^\alpha$$

with $A^0_{i\alpha} = \int_{K_0} \frac{\partial \Phi_{i_1}}{\partial X_{\alpha_1}} \frac{\partial \Phi_{i_2}}{\partial X_{\alpha_2}} \, dX$ and $G_K^\alpha = \det F_K' \frac{\partial X_{\alpha_1}}{\partial x_\beta} \frac{\partial X_{\alpha_2}}{\partial x_\beta}$

Basic usage: compiling a form

1. Implement the form using your favorite text editor (emacs):



2. Compile the form using **FFC**:

```
>> ffc poisson.form
```

This will generate C++ code (`Poisson.h`) for **DOLFIN**

Example: Classical Elasticity

FFC representation (`Elasticity.form`):

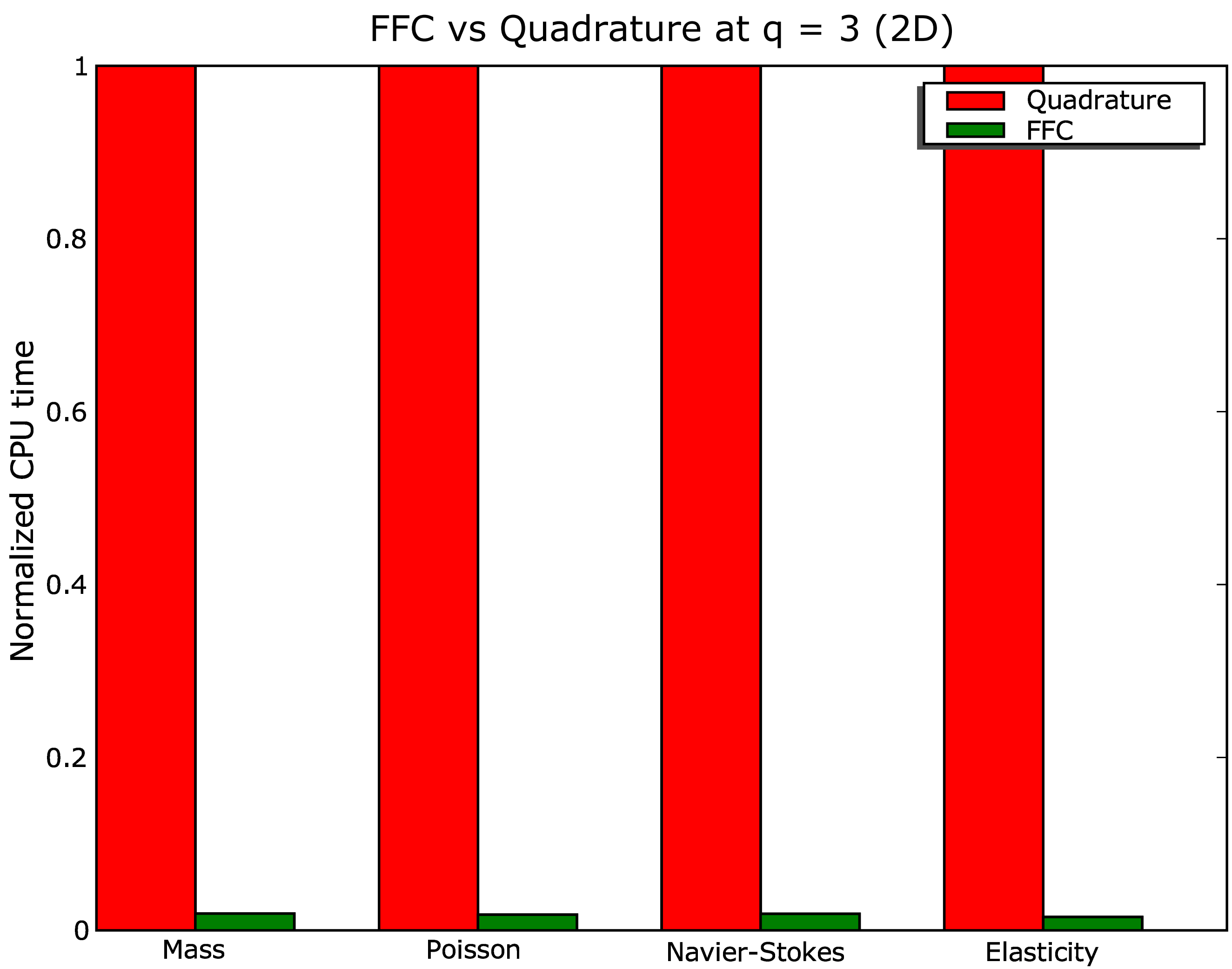
```
# The bilinear form for classical linear elasticity
# Compile this form with FFC: ffc Elasticity.form.

element = FiniteElement("Lagrange", "tetrahedron", 1)

c1 = Constant() # Lame coefficient
c2 = Constant() # Lame coefficient
f = Function(element) # Source

v = BasisFunction(element)
u = BasisFunction(element)

a = (2.0 * c1 * u[i].dx(i) * v[j].dx(j) +
          c2 * (u[i].dx(j) + u[j].dx(i)) * (v[i].dx(j) + v[j].dx(i))) * dx
L = f[i] * v[i] * dx
```

Example: Classical Elasticity

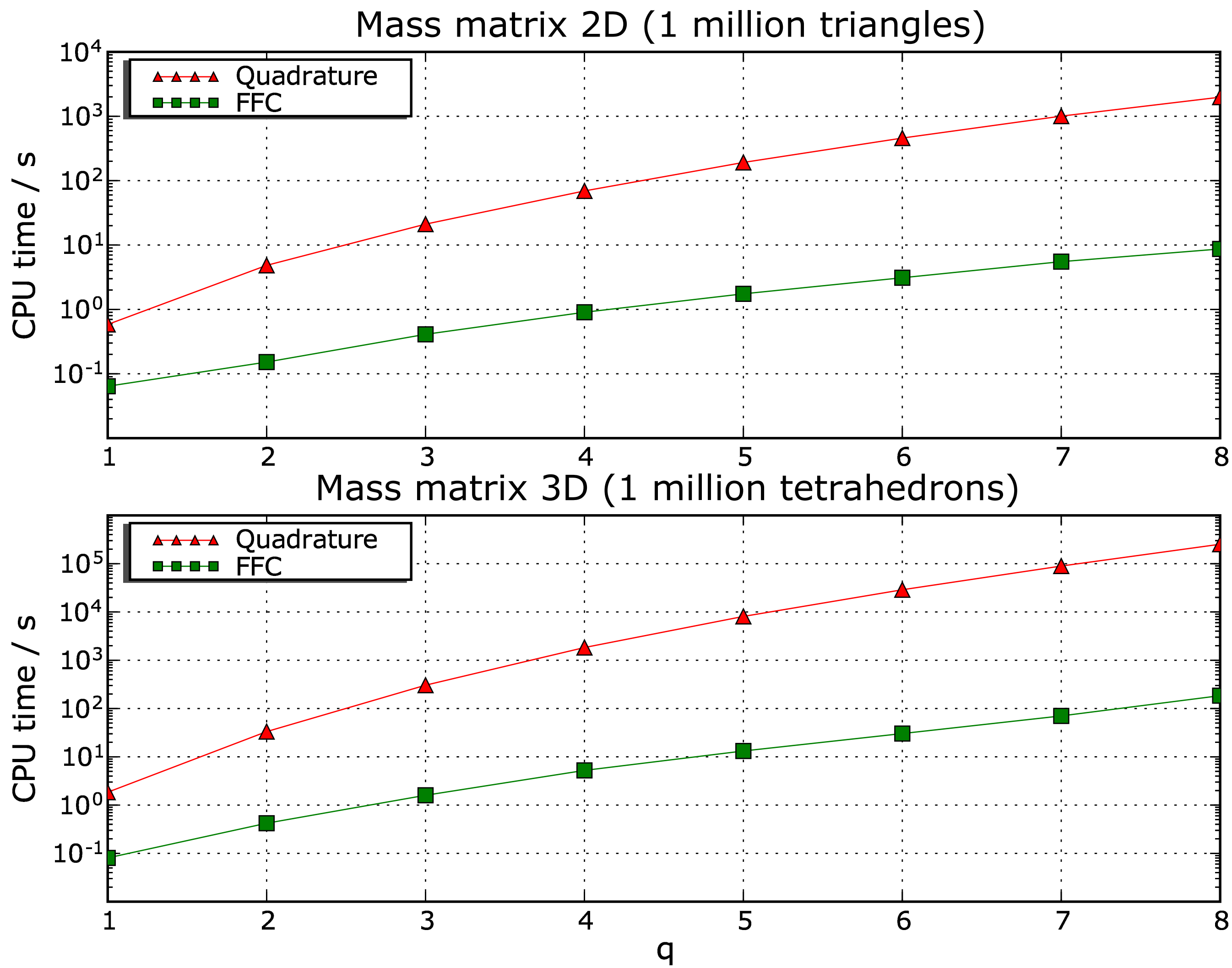
FFC output (`Elasticity.h`):

```
BilinearForm(const real& c0, const real& c1) : ...

bool interior(real* block) const
{
  // Compute geometry tensors
  real G0_0_0_0_0 = det*c0*g00*g00;
  real G0_0_0_0_1 = det*c0*g00*g10;
  ...

  // Compute element tensor
  block[0] =
  3.333333333333329e-01*G0_0_0_0_0 + 3.333333333333329e-01*G0_0_0_0_1 +
  3.333333333333329e-01*G0_0_0_0_2 + 3.333333333333329e-01*G0_0_0_1_0 +
  3.333333333333329e-01*G0_0_0_1_1 + 3.333333333333329e-01*G0_0_0_1_2 +
  3.333333333333329e-01*G0_0_0_2_0 + 3.333333333333329e-01*G0_0_0_2_1 +
  3.333333333333329e-01*G0_0_0_2_2 + 1.666666666666664e-01*G1_0_0 +
  1.666666666666664e-01*G1_0_1 + 1.666666666666664e-01*G1_0_2 +
  1.666666666666664e-01*G1_1_0 + 1.666666666666664e-01*G1_1_1 +
  ...
```

Impressive speedups

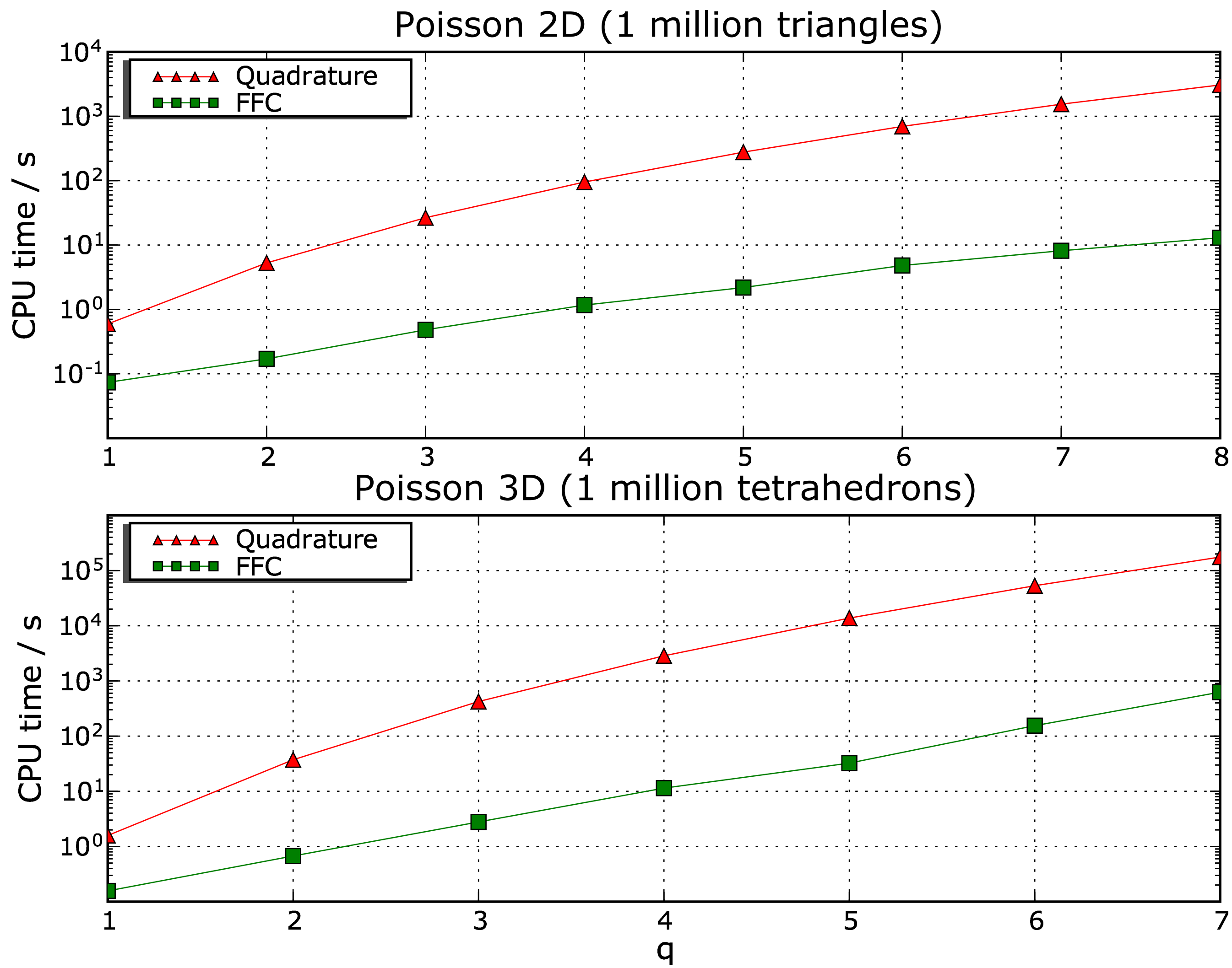


Results



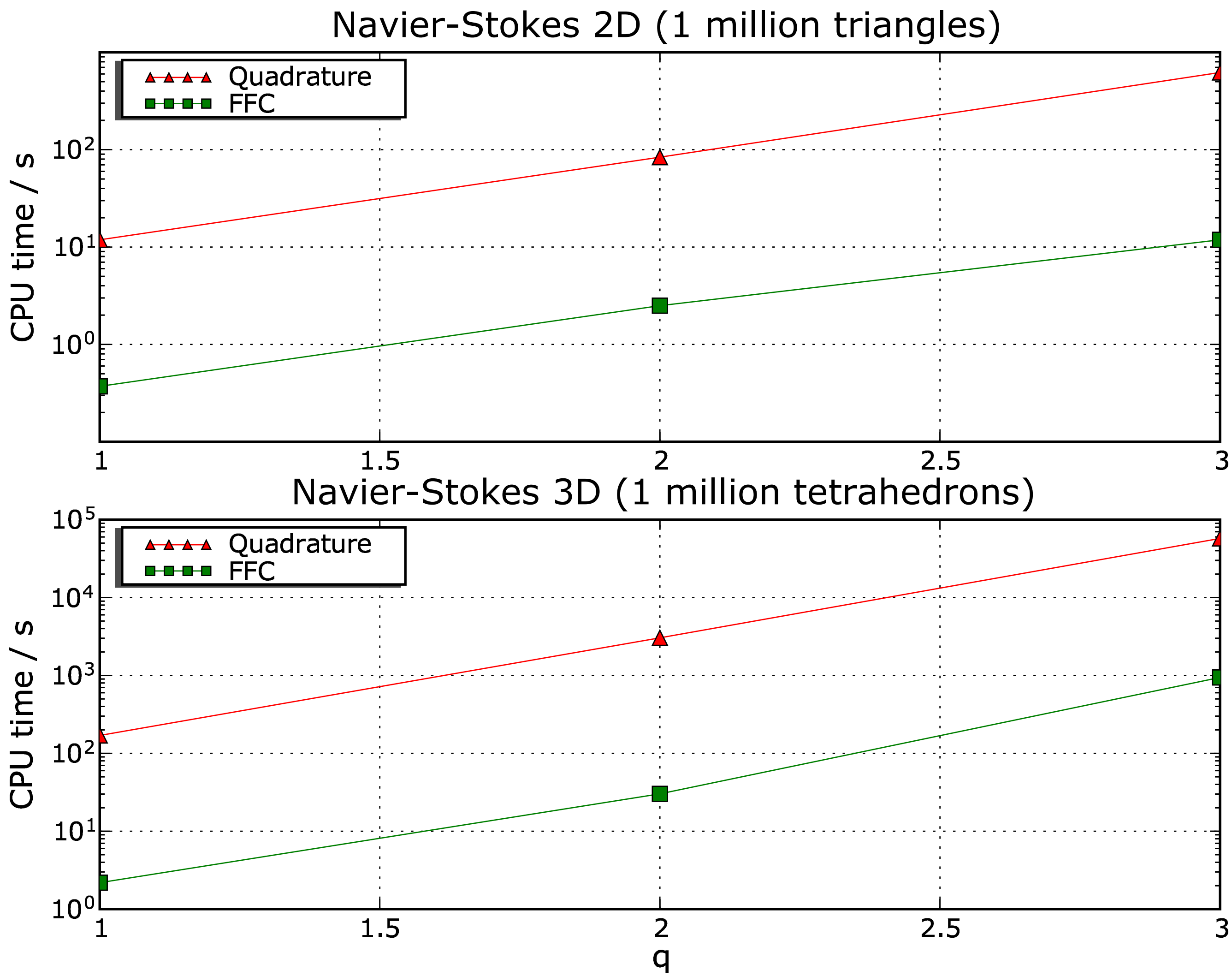
Mass matrix 2D (1 million triangles)
CPU time / s
Quadrature
FFC
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
1 2 3 4 5 6 7 8
Mass matrix 3D (1 million tetrahedrons)
CPU time / s
Quadrature
FFC
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
1 2 3 4 5 6 7 8
q

Results



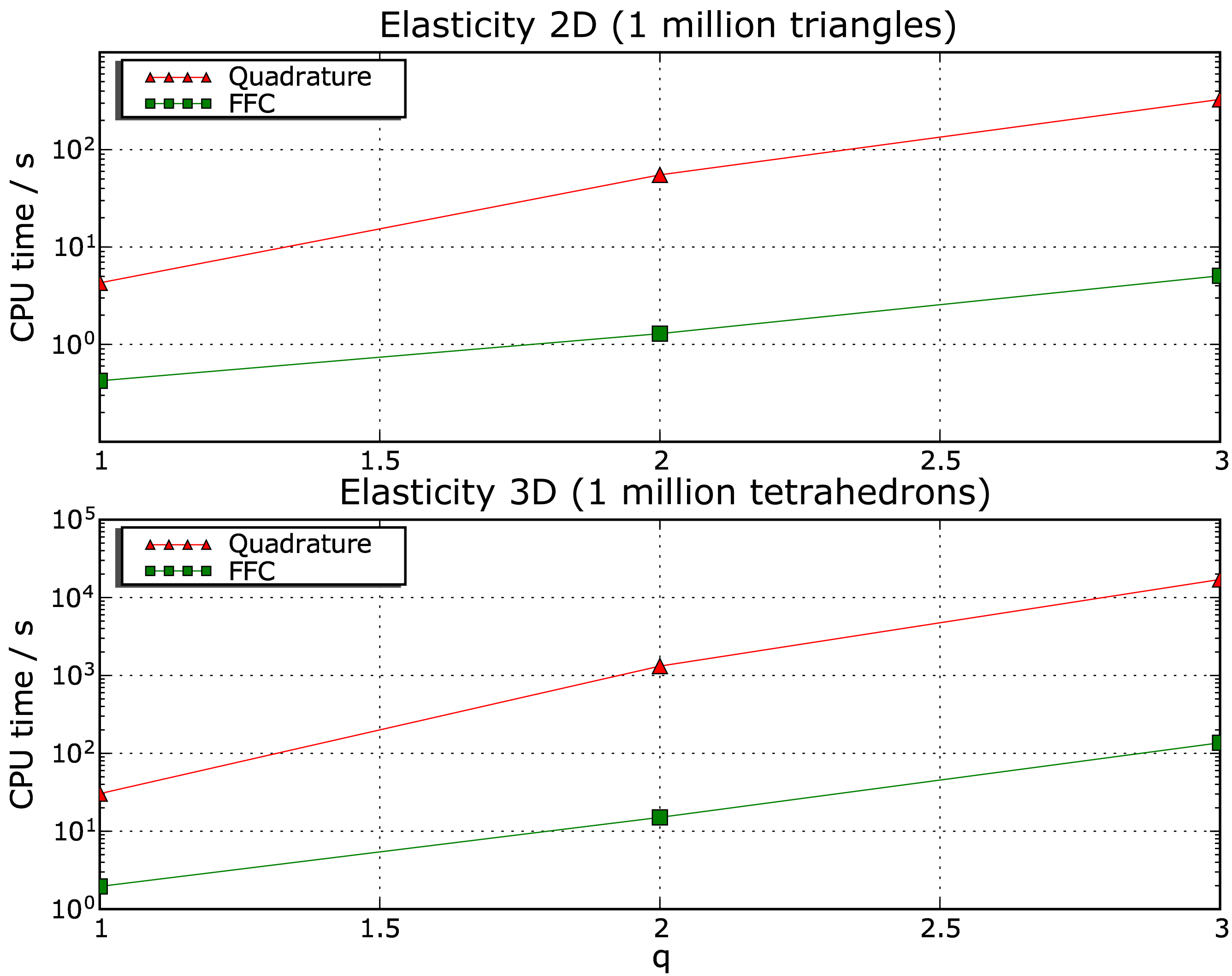
Poisson 2D (1 million triangles)
Quadrature
FFC
CPU time / s
10^4
10^3
10^2
10^1
10^0
10^-1
1
2
3
4
5
6
7
8
Poisson 3D (1 million tetrahedrons)
Quadrature
FFC
CPU time / s
10^5
10^4
10^3
10^2
10^1
10^0
1
2
3
4
5
6
7
q

Results



Navier-Stokes 2D (1 million triangles)
Quadrature
FFC
CPU time / s
10^2
10^1
10^0
1
1.5
2
2.5
3
Navier-Stokes 3D (1 million tetrahedrons)
Quadrature
FFC
CPU time / s
10^5
10^4
10^3
10^2
10^1
10^0
1
1.5
2
2.5
3
q

Results



Elasticity 2D (1 million triangles)
Quadrature
FFC
CPU time / s
10^0
10^1
10^2
1
1.5
2
2.5
3
Elasticity 3D (1 million tetrahedrons)
Quadrature
FFC
CPU time / s
10^0
10^1
10^2
10^3
10^4
10^5
1
1.5
2
2.5
3
q

Motivation for elasto-plastic model

State of the art computer games use rigid body motion with joints (Half Life 2).

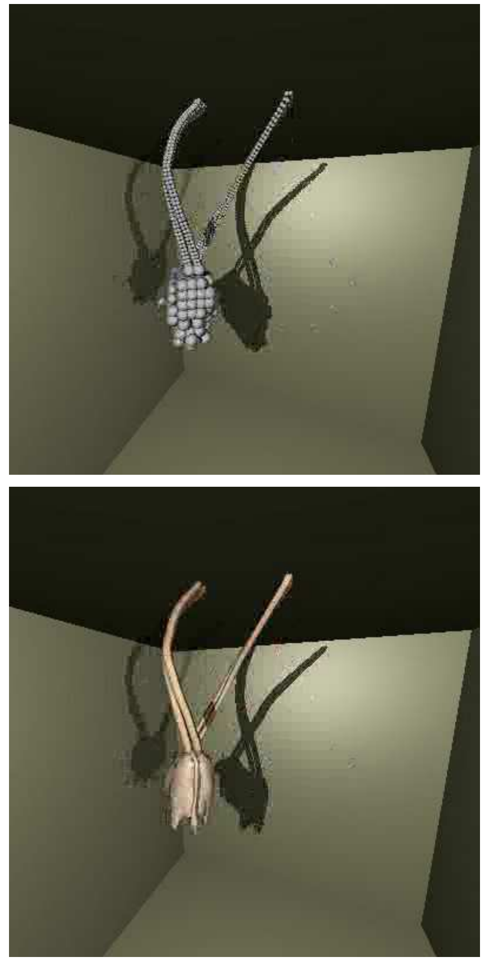
Motion pictures primarily use animation by hand, some cases of mass-spring simulation (hair, cloth).

Why don't these applications use more advanced/general models?

Traditional elasticity models are difficult to understand $\Rightarrow$ difficult to apply, use effectively.

Attempt to find a simple model, attempt to automate discretization of model.

Previous work - mass-spring model



Can we find an analogous PDE-model?

Simple derivation of model

Classical linear elasticity:

$$
\begin{aligned}
u &= x - X, \\
\dot{u} - v &= 0 \quad \text{in } \Omega^0, \\
\dot{v} - \nabla \cdot \sigma &= f \quad \text{in } \Omega^0, \\
\sigma &= E\epsilon(u) = E(\nabla u^\top + \nabla u) \\
E\epsilon &= \lambda \sum_k \epsilon_{kk} I + 2\mu\epsilon, \\
v(0,\cdot) &= v^0, \quad u(0,\cdot) = u^0 \quad \text{in } \Omega^0.
\end{aligned}
$$

Only works for small displacements. Computations carried out on fi xed geometry Ω^0. Why not use the deformed geometry $\Omega(t)$?

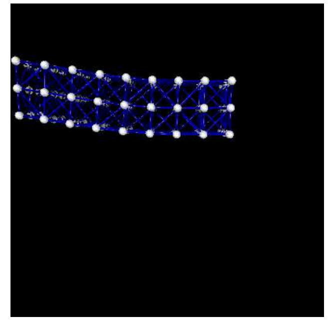
The elastic model

Formulate the model in the deformed geometry $\Omega(t)$ (updated Lagrange):

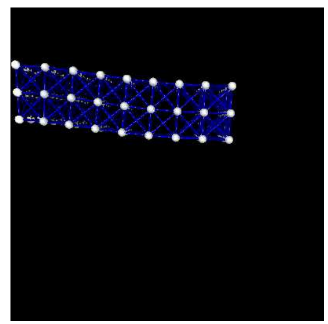
$$
\begin{aligned}
\dot{u} - v &= 0 \quad \text{in } \Omega(t), \\
\dot{v} - \nabla \cdot \sigma &= f \quad \text{in } \Omega(t), \\
\dot{\sigma} &= E\epsilon(v) = E(\nabla v^\top + \nabla v) \\
v(0,\cdot) = v^0, &\quad u(0,\cdot) = u^0 \quad \text{in } \Omega^0.
\end{aligned}
$$

The model is a piecewise linear elastic model. Given some geometry Ω_i we compute using the linear model (small displacements) for a small time step/iteration and produce the geometry Ω_{i+1}. The process is then repeated.

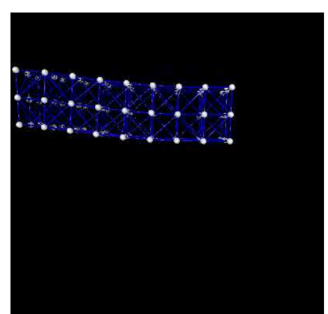
Examples



Elastic bar (Updated Lagrange)



Elastic bar (Mass-spring)



Elastic bar (Classical elasticity)

Viscosity

$$\dot{v} - \nabla \cdot \sigma - \nu \nabla \cdot \epsilon(v) = f \quad \text{in } \Omega(t)$$

We add a simple viscous term to model viscosity in materials.

Plasticity

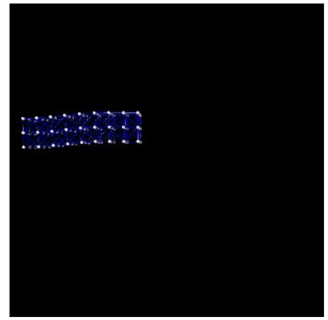
$$\dot{\sigma} = E(\epsilon(v) - \frac{1}{\nu_p}(\sigma - \pi\sigma)) \quad \text{in } \Omega(t),$$

$$\pi\sigma = \frac{\sigma}{||\sigma||}, ||\sigma|| > Y_s$$

$$\pi\sigma = \sigma, ||\sigma|| \leq Y_s$$

Visco-plastic model. $\pi\sigma$ is the projection on to the set of admissible stresses. Y_s is the yield stress of the material.

Examples (Plasticity)



Plastic bar

Implementation in FFC

FFC representation (`ElasticityUpdated.form`):

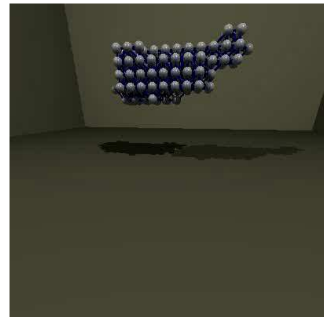
```
# Form for updated elasticity (velocity)

element1 = FiniteElement("Discontinuous vector Lagrange", "tetrahedron", 0)
element2 = FiniteElement("Vector Lagrange", "tetrahedron", 1)

nu = Constant() # viscosity coefficient

w = BasisFunction(element2)
f = Function(element2)
sigma0 = Function(element1)
epsilon0 = Function(element1)

L = (f[i] * v[i] -
(sigma0[i] * w[0].dx(i) +
sigma1[i] * w[1].dx(i) +
sigma2[i] * w[2].dx(i)) -
nu * (
epsilon0[i] * w[0].dx(i) +
epsilon1[i] * w[1].dx(i) +
epsilon2[i] * w[2].dx(i))) * dx
```

Implementation in FFC

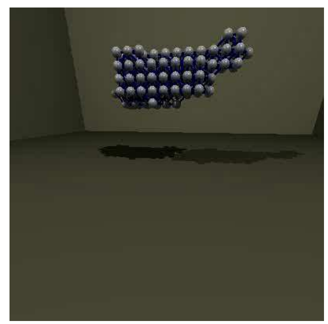
FFC representation (`ElasticityUpdatedSigma0.form`):

```
# Form for updated elasticity (stress component 0)

element1 = FiniteElement("Vector Lagrange", "tetrahedron", 1)
element2 = FiniteElement("Discontinuous vector Lagrange", "tetrahedron", 0)

c1 = Constant() # Lame coefficient
c2 = Constant() # Lame coefficient
nuplast = Constant() # Plastic viscosity

q = BasisFunction(element2)
v = Function(element1)
sigma0 = Function(element2)
sigmanorm = Function(element2) # Norm of sigma (stress)

Lplast = ((c1 * (sigma0[0] + sigma1[1] + sigma2[2]) * q[0]) +
          (c2 * sigma0[i] * q[i]))

Lelast = ((2 * c1 * v[i].dx(i) * q[0]) +
          (c2 * (v[i].dx(0) + v[0].dx(i))) * q[i])

L = (Lelast - nuplast * (1 - sigmanorm[0]) * Lplast) * dx
```

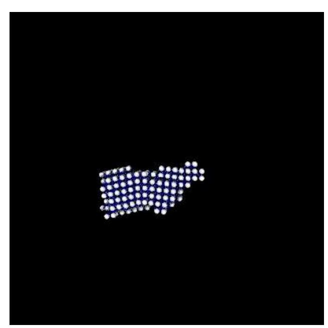
General examples



Visco-elastic cow

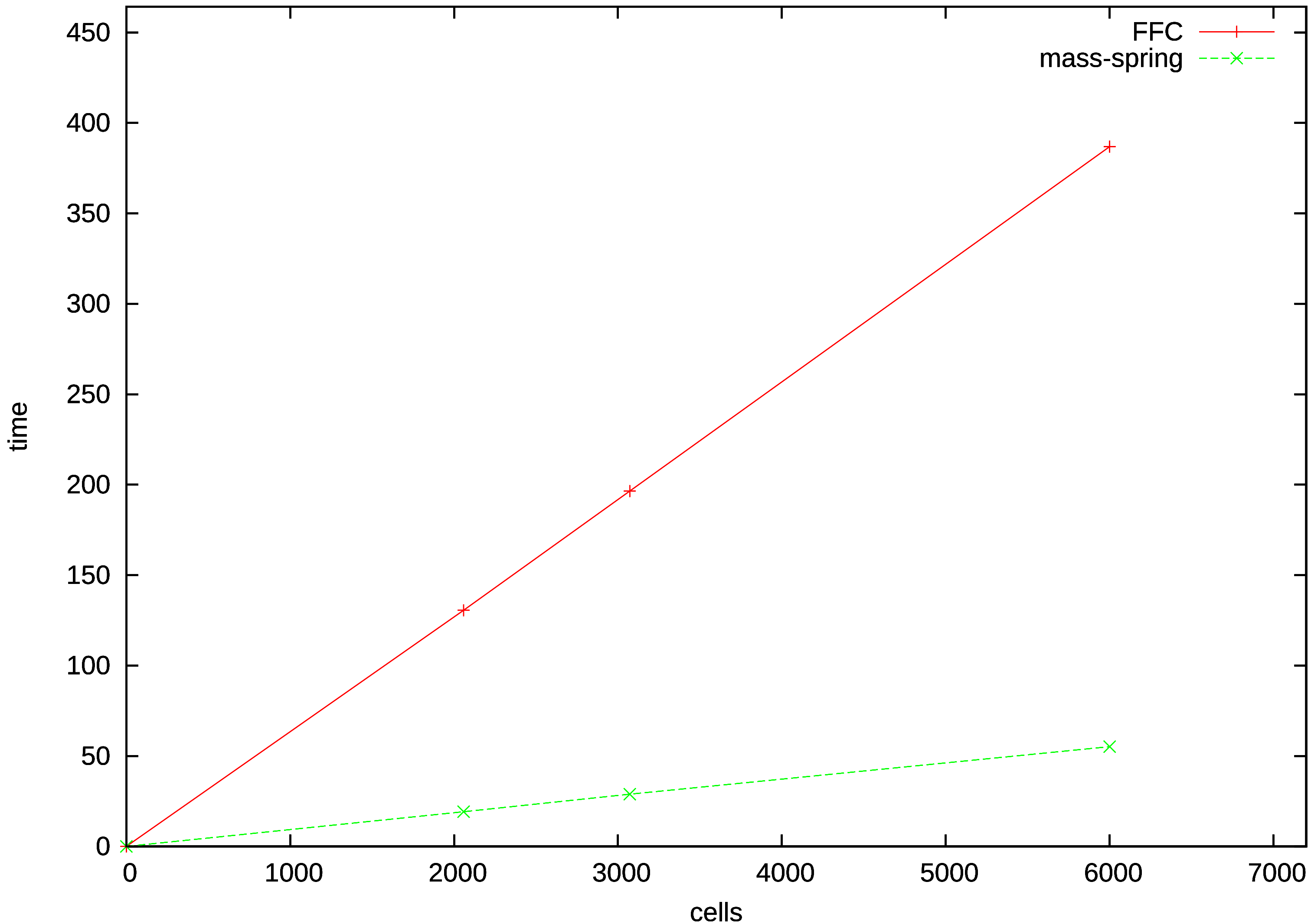


Plastic cow

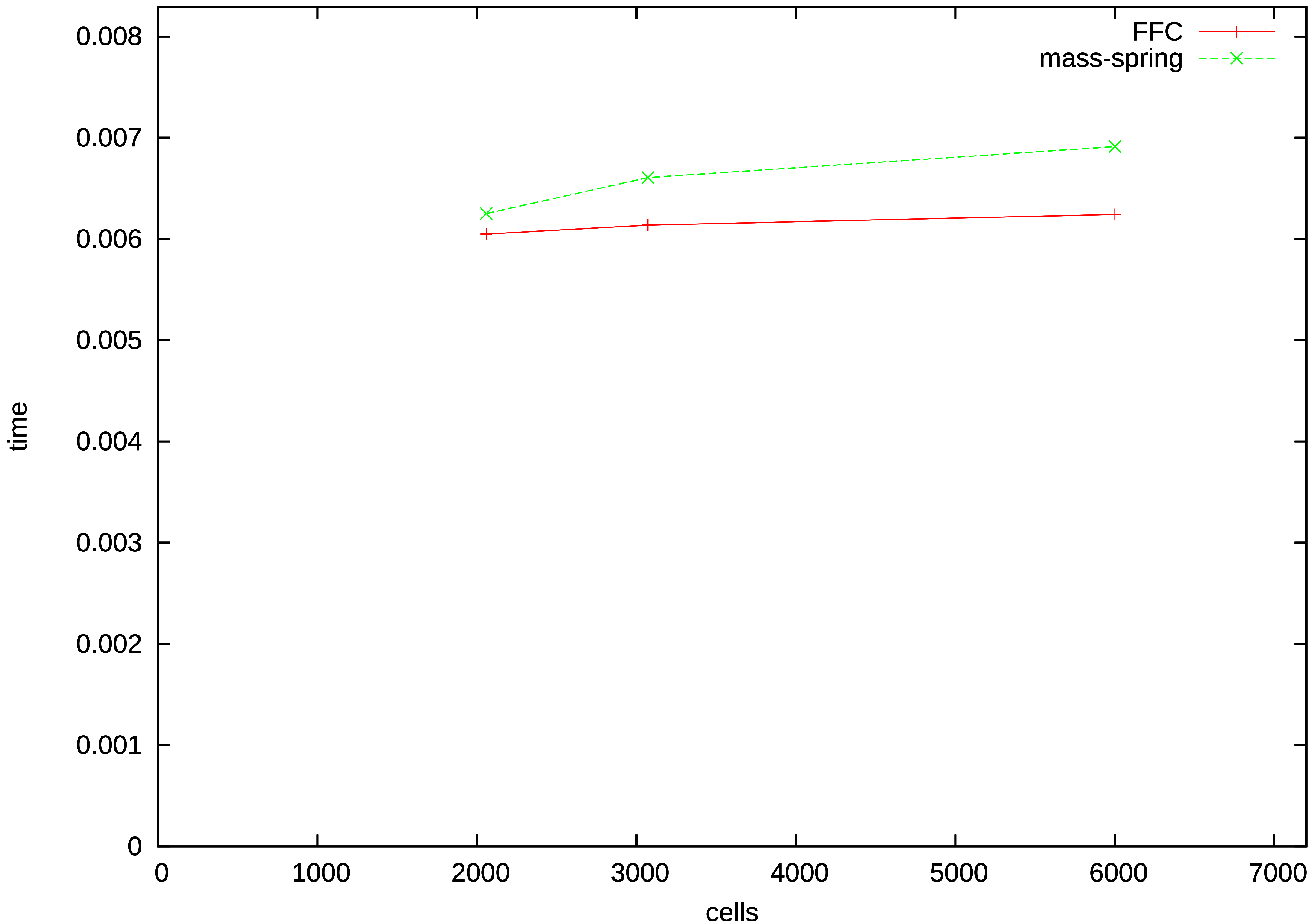


Real time simulation

Updated elasticity vs. mass-spring



Time / dof



FFC
mass-spring
time
cells
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
0
1000
2000
3000
4000
5000
6000
7000

Profiling

- Spends 90% assembling, only 10% actually evaluating form, could likely be optimized further

```
 %time    calls      name

Flat:

 15.86    226382058  dolfin::Function::interpolate()
  8.04    41162058   dolfin::AffineMap::updateTetrahedron()
  3.23    10290000   dolfin::ElasticityUpdated::LinearForm::eval()
  3.23    740882058  dolfin::Cell::id() const
  2.57    617498784  dolfin::GenericCell::nodeID() const
  2.46    10290000   dolfin::ElasticityUpdatedSigma2::LinearForm::eval()
  2.30    10290000   dolfin::ElasticityUpdatedSigma0::LinearForm::eval()
  2.15    10290000   dolfin::ElasticityUpdatedSigma1::LinearForm::eval()

Graph:

 89.7     20000      dolfin::FEM::assemble()
 49.8     41162058   dolfin::Form::updateCoefficients()
```

Future work

FFC:

- Independent comparisons for FFC - benchmark against other PDE packages (also finite difference packages).
- Extend the elastic model: contact, friction (mass-spring model already does this).
- Space adaptivity
- Apply model in real applications (games for instance).
- Interface to fluid mechanics (Navier-Stokes).