



# FEniCS Course

## Lecture 8: The Stokes problem

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*Contributors*

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# The Stokes equations

$$-\Delta u + \nabla p = f \quad \text{in } \Omega \quad \text{Momentum equation}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \quad \text{Continuity equation}$$

$$u = g_D \quad \text{on } \partial\Omega_D$$

$$\frac{\partial u}{\partial n} - pn = g_N \quad \text{on } \partial\Omega_N$$

- $u$  is the fluid velocity and  $p$  is the pressure
- $f$  is a given body force per unit volume
- $g_D$  is a given boundary flow
- $g_N$  is a given function for the natural boundary condition

## Variational problem

Multiply the momentum equation by a test function  $v$  and integrate by parts:

$$\int_{\Omega} \nabla u : \nabla v \, dx - \int_{\Omega} p \nabla \cdot v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\partial\Omega_N} g_N \cdot v \, ds$$

Short-hand notation:

$$\underbrace{\langle \nabla u, \nabla v \rangle}_{a(u,v)} - \underbrace{\langle p, \nabla \cdot v \rangle}_{b(v,p)} = \underbrace{\langle f, v \rangle + \langle g_N, v \rangle_{\partial\Omega_N}}_{L(v)}$$

Multiply the continuity equation by a test function  $q$ :

$$\underbrace{\pm \langle \nabla \cdot u, q \rangle}_{b(u,q)} = 0$$

Definitions of  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  are meaningful if  $u \in H^1(\Omega)$  and  $p \in L^2(\Omega)$

## Saddle point formulation of the Stokes problem

The Stokes problem is an example of a **saddle point problem**:

Find  $(u, p) \in V \times Q$  such that for all  $(v, q) \in \widehat{V} \times \widehat{Q}$

$$a(u, v) + b(v, p) = L(v)$$

$$b(u, q) = 0$$

Sum up:  $A(u, p; v, q) := a(u, v) + b(v, p) + b(u, q) = L(v)$

**Mixed spaces:**

$$V = [H_{gD, \Gamma_D}^1(\Omega)]^d$$

$$\widehat{V} = [H_{0, \Gamma_D}^1(\Omega)]^d$$

$$Q = L^2(\Omega)$$

$$\widehat{Q} = L^2(\Omega)$$

The **inf-sup condition**

$$\inf_{q \in Q} \sup_{v \in \widehat{V}} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq C$$

is critical for the unique solvability of the saddle point problem

# Discrete variational problem

Find  $(u_h, p_h) \in V_h \times Q_h$  such that for all  $(v_h, q_h) \in \widehat{V}_h \times \widehat{Q}_h$

$$A_h(u_h, p_h; v_h, q_h) := a_h(u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = L_h(v_h)$$

A **stable mixed element**  $V_h \times Q_h \subset V \times Q$  should satisfy a uniform **inf-sup condition**

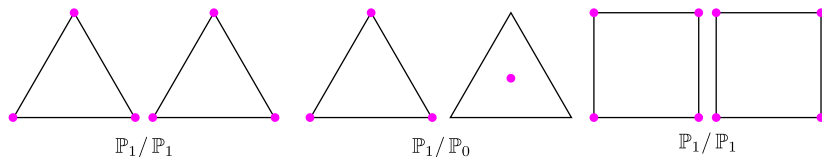
$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq c_b$$

with  $c_b$  independent of the mesh  $\mathcal{T}_h$ !

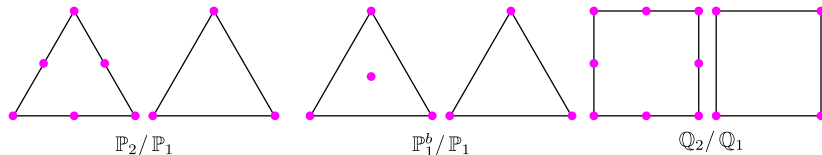
$\Rightarrow$  The right “mixture” of elements is **critical** for stability and convergence!

# Unstable and stable Stokes elements

## Unstable elements



## Stable elements



Taylor-Hood elements:  $\mathbb{P}_{k+1}/\mathbb{P}_k, \mathbb{Q}_{k+1}/\mathbb{Q}_k$  for  $k \geq 1$

Mini-element:  $\mathbb{P}_1^b/\mathbb{P}_1$

# Useful FEniCS tools (I)

Mixed elements:

```
V = VectorFunctionSpace(mesh, "Lagrange", 2)
Q = FunctionSpace(mesh, "Lagrange", 1)
W = V*Q
```

Defining functions, test and trial functions:

```
up = Function(W)
(u,p) = split(up)
```

Shortcut:

```
(u, p) = Functions(W)
# similar for test and trial functions
(u, p) = TrialFunctions(W)
(v, q) = TestFunctions(W)
```

## Useful FEniCS tools (II)

Access subspaces:

```
W.sub(0) #corresponds to V  
W.sub(1) #corresponds to Q
```

Splitting solution into components:

```
w = Function(W)  
solve(a == L, w, bcs)  
(u, p) = w.split()
```

Rectangle mesh:

```
mesh = RectangleMesh(0.0, 0.0, 5.0, 1.0, 50, 10)
```

```
h = CellSize(mesh)
```



## Exercise: Spurious pressure modes

Compute the finite element approximation for Couette flow on the unit square. Use the boundary data

$$u = 1 \text{ on } y = 1, \quad u = 0 \text{ on } y = 0, \quad g_N = 0 \text{ on } x = 0 \text{ or } x = 1$$

Use  $\mathbb{P}_1/\mathbb{P}_1$  and  $\mathbb{P}_1/\mathbb{P}_0$  elements. The exact solution is given by

$$u = (y, 0), \quad p = 0$$

What do you observe? Why?

## Exercise: A stabilized $\mathbb{P}_1/\mathbb{P}_1$ method

Define the bilinear forms

$$a_h(u_h, v_h) = (\nabla u_h, \nabla v_h)$$

$$b_h(v_h, q_h) = -(\nabla \cdot v_h, q_h)$$

$$c_h(p_h, q_h) = \sum_{T \in \mathcal{T}_h} \mu_T (\nabla p_h, \nabla q_h)$$

and solve: find  $(u_h, p_h) \in V_h \times Q_h$  such that  $\forall (v_h, q_h) \in \widehat{V}_h \times \widehat{Q}_h$

$$\begin{aligned} A(u_h, p_h; v_h, q_h) &:= a(u_h, v_h) + b(v_h, p_h) + b(u_h, q_h) - c(p_h, q_h) \\ &= (f, v_h) - \sum_{T \in \mathcal{T}_h} \mu_T (f, \nabla q_h) \end{aligned}$$

**Exercise:** Implement this scheme for the Couette flow example using  $\mu_T = \beta h_T^2$ ,  $\beta = 0.2$ .