FEniCS Course

Lecture 2: Static linear PDEs

Contributors
Hans Petter Langtangen
Anders Logg
André Massing

Hello World!

We will solve Poisson's equation, the Hello World of scientific computing:

$$\begin{aligned} -\Delta u &= f \quad &&\text{in } \Omega \\ u &= u_0 \quad &&\text{on } \partial\Omega \end{aligned}$$

Poisson's equation arises in numerous contexts:

- heat conduction, electrostatics, diffusion of substances, twisting of elastic rods, inviscid fluid flow, water waves, magnetostatics
- as part of numerical splitting strategies of more complicated systems of PDEs, in particular the Navier–Stokes equations

The FEM cookbook

(i) $\mathcal{A}u = f$ — *Partial differential equation*

Multiply by v

(ii) $a(u, v) = L(v)$ — *Continuous variational problem*

Take $V_h \subset V$

(iii) $a(u_h, v) = L(v)$ — *Discrete variational problem*

Let $u_h = \sum_j U_j \phi_j$

(iv) $AU = b$ — *System of discrete equations*

Solving PDEs in FEniCS

Solving a physical problem with FEniCS consists of the following steps:

1. Identify the PDE and its boundary conditions
2. Reformulate the PDE problem as a variational problem
3. Make a Python program where the formulas in the variational problem are coded, along with definitions of input data such as f, u_0, and a mesh for Ω
4. Add statements in the program for solving the variational problem, computing derived quantities such as ∇u, and visualizing the results

Deriving a variational problem for Poisson's equation

The simple recipe is: multiply the PDE by a test function v and integrate over Ω:

$$-\int_\Omega (\Delta u) v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Then integrate by parts and set $v = 0$ on the Dirichlet boundary:

$$-\int_\Omega (\Delta u) v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s}_{=0}$$

We find that:

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Variational problem for Poisson's equation

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x$$

for all $v \in \hat{V}$

The trial space V and the test space $\hat{V}$ are (here) given by

$$V = \{v \in H^1(\Omega) : v = u_0 \text{ on } \partial\Omega\}$$
$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$$

Discrete variational problem for Poisson's equation

We approximate the continuous variational problem with a discrete variational problem posed on finite dimensional subspaces of V and $\hat{V}$:

$$
\begin{aligned}
V_h &\subset V \\
\hat{V}_h &\subset \hat{V}
\end{aligned}
$$

Find $u_h \in V_h \subset V$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x$$

for all $v \in \hat{V}_h \subset \hat{V}$

Canonical variational problem

The following canonical notation is used in FEniCS: find $u \in V$ such that

$$a(u, v) = L(v)$$

for all $v \in \hat{V}$

For Poisson's equation, we have

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}x \\ L(v) &= \int_{\Omega} f v \, \mathrm{d}x \end{aligned}$$

$a(u, v)$ is a *bilinear form* and $L(v)$ is a *linear form*

Poisson example 1

Strong form

Let $\Omega = [0,1] \times [0,1]$. Solve

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Weak form

Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega 1 v \, \mathrm{d}x}_{L(v)}$$

Poisson example 2

- Domain:

$$\Omega = [0,1] \times [0,1]$$
$$\partial\Omega_D = \{0\} \times [0,1] \cup \{1\} \times [0,1]$$
$$\partial\Omega_N = [0,1] \times \{0\} \cup [0,1] \times \{1\}$$

- Source and boundary values:

$$f(x,y) = 2\cos(2\pi x)\cos(2\pi y)$$
$$g_D(x,y) = 0.1\cos(2\pi y)$$

Strong form

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = g_D \quad \text{on } \partial\Omega_D$$
$$\frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega_N$$

Weak form

Find $u \in V$ such that for all $v \in \widehat{V}$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{L(v)}$$

- Function spaces:

$$V = \{v \in H^1(\Omega) : v = g_D \text{ on } \partial\Omega_D\}$$
$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$$

Poisson example 2

- Domain:

$$\begin{aligned}\Omega &= [0,1] \times [0,1] \\ \partial\Omega_D &= \{0\} \times [0,1] \cup \{1\} \times [0,1] \\ \partial\Omega_N &= [0,1] \times \{0\} \cup [0,1] \times \{1\}\end{aligned}$$

- Source and boundary values:

$$\begin{aligned}f(x,y) &= 2\cos(2\pi x)\cos(2\pi y) \\ g_D(x,y) &= 0.1\cos(2\pi y)\end{aligned}$$

Strong form

$$\begin{aligned}-\Delta u &= f \quad \text{in } \Omega \\ u &= g_D \quad \text{on } \partial\Omega_D \\ \frac{\partial u}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \partial\Omega_N\end{aligned}$$

Weak form

Find $u \in V$ such that for all $v \in \widehat{V}$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{L(v)}$$

- Function spaces:

$$\begin{aligned}V &= \{v \in H^1(\Omega) : v = g_D \text{ on } \partial\Omega_D\} \\ \hat{V} &= \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}\end{aligned}$$

Poisson example 3

- Domain:

$$\Omega = [0,1] \times [0,1] \setminus \text{dolphin domain}$$
$$\partial\Omega_D = \{0\} \times [0,1] \cup \{1\} \times [0,1]$$
$$\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$$

- Source and boundary values:

$$f(x,y) = 2\cos(2\pi x)\cos(2\pi y)$$
$$g_D(x,y) = 0.5\cos(2\pi y) \quad \text{on } x = 0$$
$$g_D(x,y) = 1 \quad \text{on } x = 1$$
$$g_N(x,y) = \sin(\pi x)\sin(\pi y)$$

Strong form

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = g_D \quad \text{on } \partial\Omega_D$$
$$-\frac{\partial u}{\partial \boldsymbol{n}} = g_N \quad \text{on } \partial\Omega_N$$

Weak form

Find $u \in V$ such that for all $v \in \widehat{V}$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x + \int_{\partial\Omega_N} g v \,\mathrm{d}s}_{L(v)}$$

- Function spaces:

$$V = \{v \in H^1(\Omega) : v = g_D \text{ on } \partial\Omega_D\}$$
$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$$

Poisson example 3

- Domain:

$$\begin{aligned} \Omega &= [0,1] \times [0,1] \setminus \text{dolphin domain} \\ \partial\Omega_D &= \{0\} \times [0,1] \cup \{1\} \times [0,1] \\ \partial\Omega_N &= \partial\Omega \setminus \partial\Omega_D \end{aligned}$$

- Source and boundary values:

$$\begin{aligned} f(x,y) &= 2\cos(2\pi x)\cos(2\pi y) \\ g_D(x,y) &= 0.5\cos(2\pi y) \quad \text{on } x = 0 \\ g_D(x,y) &= 1 \quad \text{on } x = 1 \\ g_N(x,y) &= \sin(\pi x)\sin(\pi y) \end{aligned}$$

Strong form

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g_D \quad \text{on } \partial\Omega_D \\ -\frac{\partial u}{\partial \boldsymbol{n}} &= g_N \quad \text{on } \partial\Omega_N \end{aligned}$$

Weak form

Find $u \in V$ such that for all $v \in \widehat{V}$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x + \int_{\partial\Omega_N} g v \,\mathrm{d}s}_{L(v)}$$

- Function spaces:

$$\begin{aligned} V &= \{v \in H^1(\Omega) : v = g_D \text{ on } \partial\Omega_D\} \\ \hat{V} &= \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\} \end{aligned}$$

Poisson example 3: Mission possible

Your mission

- open and plot the dolfin mesh saved in dolfin-channel.xml
- solve the discrete variational problem
- export the solution to a pvd file and visualize it in Paraview

Your tools

Read in a mesh

```
mesh = Mesh("dolfin-channel.xml")
```

Inhomogeneus Neuman boundary condition

```
L = ... + g_N*v*ds
```

List of Dirchlet boundary conditions

```
            bc0 = DirichletBC(...)
            bc1 = DirichletBC(...)
            bcs = [bc0, bc1]
```

Save solution in VTK format

```
u_file = File("poisson_3.pvd")
u_file << u
```

Poisson example 3: Extra mission

- Choose a variable conductivity of the form

$$k(x, y) = 1 + e^{(x^2+y^2)}$$

- What is the expression of the heat flux σ across the boundary now (opposed to $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \frac{\partial u}{\partial \boldsymbol{n}}$ in the original problem)?
- Replace the inhomogeneus Neumann boundary condition by a Robin boundary condition

$$-\sigma \cdot \boldsymbol{n} = u - g_N \quad \text{on } \partial\Omega_N$$

- Solve

$$\begin{aligned} -\nabla \cdot (k(x, y)\nabla u) &= f \quad \text{in } \Omega \\ u &= g_D \quad \text{on } \partial\Omega_D \\ -\sigma \cdot \boldsymbol{n} &= u - g_N \quad \text{on } \partial\Omega_N \end{aligned}$$

  by finding the weak formulation of the problem and solving it using `FEniCS`

Poisson example 4

- Domain:

$$\begin{aligned}\Omega_1 &= [0,1]\times[0,0.5]\\ \Omega_2 &= [0,1]\times[0.5,1]\\ \Omega &= \Omega_1\cup\Omega_2\\ \partial\Omega_D &= \partial\Omega\end{aligned}$$

- Conductivity, source and boundary values:

$$\begin{aligned}k(x,y) &= \begin{cases}10 & \text{in } \Omega_1\\ 50+e^{50(0.5-y)^2} & \text{in } \Omega_2\end{cases}\\ f(x,y) &= 1\\ g_D(x,y) &= 0\end{aligned}$$

Strong form

$$\begin{aligned}-\nabla\cdot(k_1(x,y)\nabla u) &= f \quad \text{in } \Omega_1\\ -\nabla\cdot(k_2(x,y)\nabla u)+u &= f \quad \text{in } \Omega_2\\ u &= g_D \quad \text{on } \partial\Omega_D\end{aligned}$$

Weak form

Find $u\in V$ such that for all $v\in\widehat{V}$

$$\underbrace{\int_{\Omega_1} k_1\nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\Omega_2} k_2\nabla u\cdot\nabla v+uv\,\mathrm{d}x}_{a(u,v)}=\underbrace{\int_{\Omega} fv\,\mathrm{d}x}_{L(v)}$$

- Function spaces:

$$\begin{aligned}V &= \{v\in H^1(\Omega): v=g_D \text{ on } \partial\Omega_D\}\\ \hat{V} &= \{v\in H^1(\Omega): v=0 \text{ on } \partial\Omega_D\}\end{aligned}$$

Poisson example 4

- Domain:

$$\begin{aligned}\Omega_1 &= [0,1]\times[0,0.5]\\ \Omega_2 &= [0,1]\times[0.5,1]\\ \Omega &= \Omega_1\cup\Omega_2\\ \partial\Omega_D &= \partial\Omega\end{aligned}$$

- Conductivity, source and boundary values:

$$\begin{aligned}k(x,y) &= \begin{cases}10 & \text{in } \Omega_1\\ 50+e^{50(0.5-y)^2} & \text{in } \Omega_2\end{cases}\\ f(x,y) &= 1\\ g_D(x,y) &= 0\end{aligned}$$

Strong form

$$\begin{aligned}-\nabla\cdot(k_1(x,y)\nabla u) &= f \quad \text{in } \Omega_1\\ -\nabla\cdot(k_2(x,y)\nabla u)+u &= f \quad \text{in } \Omega_2\\ u &= g_D \quad \text{on } \partial\Omega_D\end{aligned}$$

Weak form

Find $u\in V$ such that for all $v\in\widehat{V}$

$$\underbrace{\int_{\Omega_1} k_1\nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\Omega_2} k_2\nabla u\cdot\nabla v+uv\,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_{\Omega} fv\,\mathrm{d}x}_{L(v)}$$

- Function spaces:

$$\begin{aligned}V &= \{v\in H^1(\Omega): v=g_D \text{ on } \partial\Omega_D\}\\ \hat{V} &= \{v\in H^1(\Omega): v=0 \text{ on } \partial\Omega_D\}\end{aligned}$$

The FEniCS challenge!

- Domain:

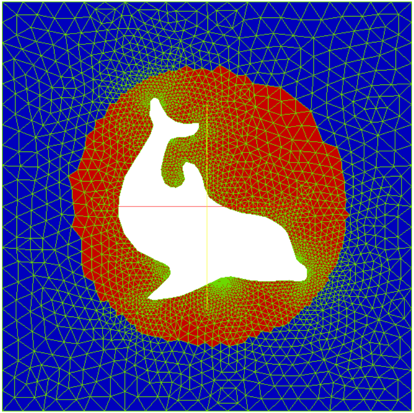
$$\begin{aligned}
\Omega_{DO} &= \text{dolphin domain} \\
\Omega &= [0,1] \times [0,1] \setminus \Omega_{DO} \\
\Omega_1 &= \{T \in \mathcal{T} : T \subset B_{0.35}(0.5, 0.5)\} \\
\Omega_2 &= \Omega \setminus \Omega_1 \\
\partial\Omega_D &= \{0\} \times [0,1] \cup \{1\} \times [0,1] \\
\partial\Omega_{N,1} &= \partial\Omega_{DO} \\
\partial\Omega_{N,2} &= [0,1] \times \{0\} \cup [0,1] \times \{1\}
\end{aligned}$$

- Conductivity, source and boundary values:

$$\begin{aligned}
k(x,y) &= \begin{cases} 10 & \text{in } \Omega_1 \\ 50 + e^{50(0.5-y)^2} & \text{in } \Omega_2 \end{cases} \\
f(x,y) &= 1 \\
g_D(x,y) &= 0 \\
g_{N,1}(x,y) &= 0 \\
g_{N,2}(x,y) &= \sin(\pi x)\sin(\pi y)
\end{aligned}$$

- As an alternative, reuse the source function and the Dirichlet boundary values from exercise 3:

$$\begin{aligned}
f(x,y) &= 2\cos(2\pi x)\cos(2\pi y) \\
g_D(x,y) &= 0.5\cos(2\pi y) \quad \text{on } x = 0 \\
g_D(x,y) &= 1 \quad \text{on } x = 1
\end{aligned}$$



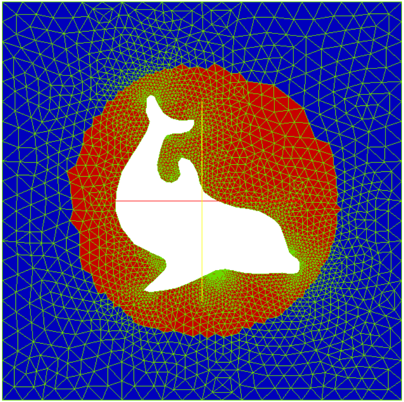
The FEniCS challenge!

Solve

$$-\nabla \cdot (k_1(x,y)\nabla u) + u = f \quad \text{in } \Omega_1$$
$$-\nabla \cdot (k_2(x,y)\nabla u) = f \quad \text{in } \Omega_2$$

$$u = g_D \quad \text{on } \partial\Omega_D$$
$$-\frac{\partial u}{\partial \boldsymbol{n}} = g_{N,1} \quad \text{on } \partial\Omega_{N,1}$$
$$-\frac{\partial u}{\partial \boldsymbol{n}} = u - g_{N,2} \quad \text{on } \partial\Omega_{N,2}$$

by first finding the weak formulation and then solving the system numerically using FEniCS



Tools

Define facet markers

```
boundary_markers = FacetFunction("size_t",mesh)
...
```

A redefinition of “ds” is necessary as well (why?). How will that probably look like?