FEniCS Course

Lecture 10: Discontinuous Galerkin methods for elliptic equations

Contributors
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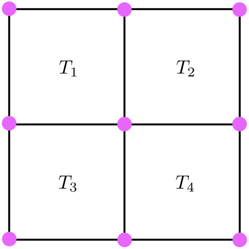
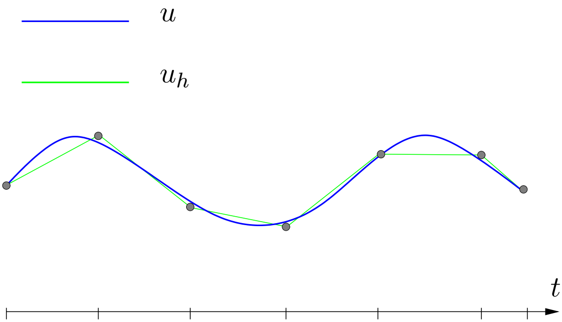
The discontinuous Galerkin (DG) method uses discontinuous basis functions





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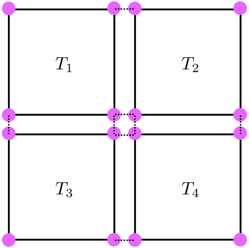

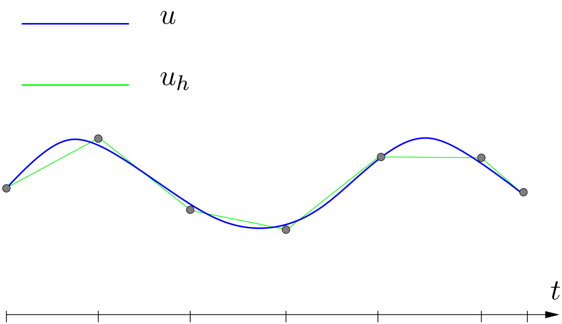

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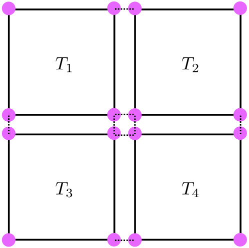

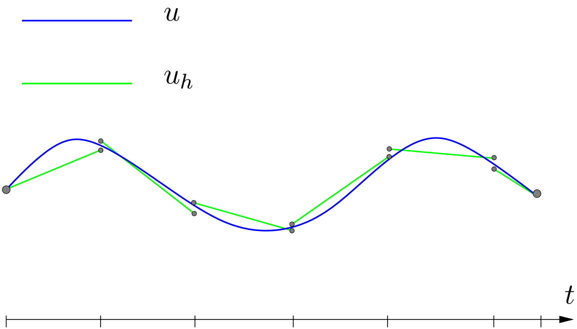

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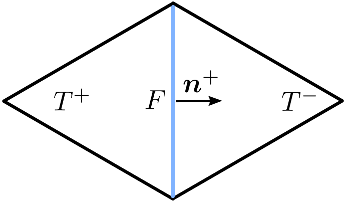
Discontinuous Galerkin: notation



Average of a scalar field:
$\langle v \rangle = \frac{1}{2}(v^+ + v^-)$

Jump of a scalar field:
$[\![v]\!] = (v^+ - v^-)n$

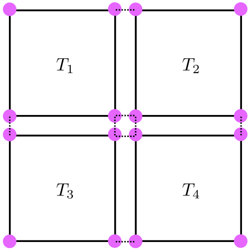
Average of a vector field:
$\langle B \rangle = \frac{1}{2}(B^+ + B^-)$

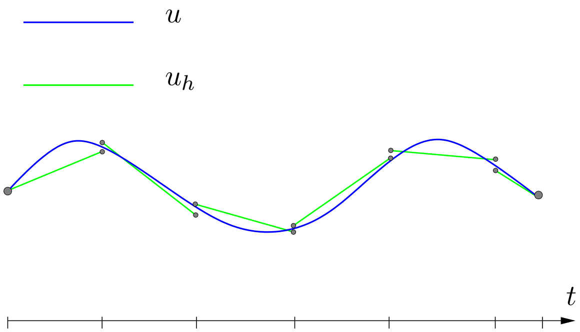
Jump of a vector field:
$[\![B]\!] = (B^+ - B^-) \cdot n$

Jump identity (Exercise for the reader!)

$$[\![Bv]\!] = [\![B]\!] \langle v \rangle + \langle B \rangle [\![v]\!]$$

The DG method eases mesh (h-) adaptivity





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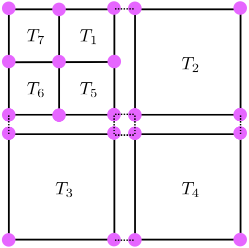

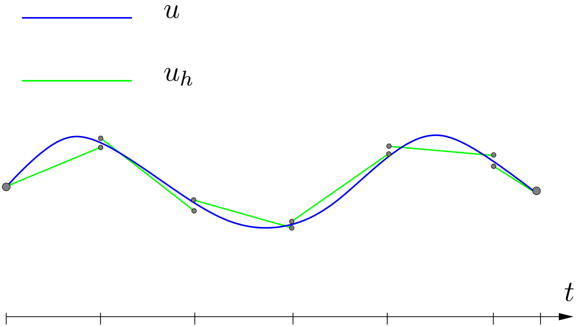

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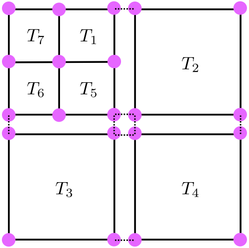
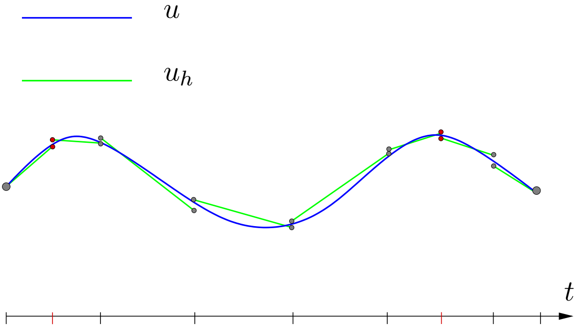
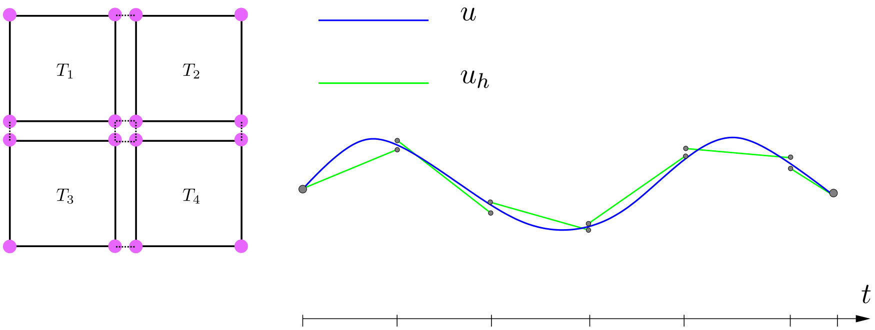




The DG method eases space (p-) adaptivity



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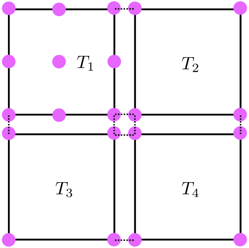

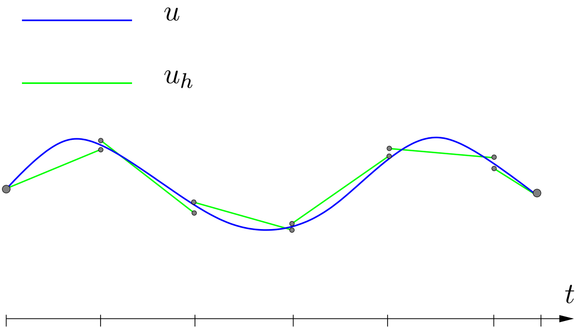

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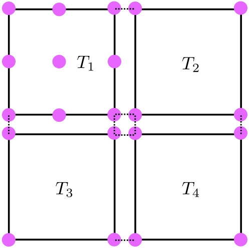
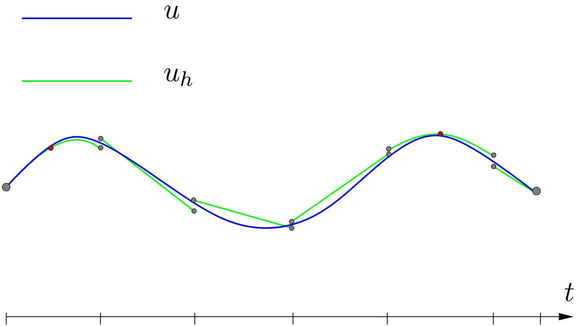




Poisson's equation revisited

Consider Poisson's equation again, now with homogeneous Dirichlet boundary conditions for simplicity

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assume that we have a mesh $\mathcal{T} = \{T\}$ of Ω

Let's also say that we would like the solution u and its flux $\operatorname{grad} u \cdot n$, where n is the facet normal, to be continuous across all facets of the mesh.

We are going to derive a discontinuous Galerkin (DG) formulation for this equations.

Deriving a DG formulation (i)

Multiply by a function v and integrate over Ω.

$$\int_\Omega -\Delta u v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Integrate by parts? No, wait for it!

Assume that you have a mesh $\mathcal{T}$ of Ω with cells $\{T\}$ and split left integral into sum over cell integrals:

$$\sum_{T \in \mathcal{T}} \int_T -\Delta u v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Now integrate by parts!

$$\sum_{T \in \mathcal{T}} \int_T \operatorname{grad} u \cdot \operatorname{grad} v \,\mathrm{d}x - \sum_{T \in \mathcal{T}} \int_{\partial T} \operatorname{grad} u \cdot n \, v \,\mathrm{d}s = \int_\Omega f v \,\mathrm{d}x$$

Deriving a DG formulation (ii)

Each interior facet e is shared by two cells (T^+ and T^-). We denote the set of all interior facets by $\mathcal{F}_i$ and the set of all exterior (boundary) facets by $\mathcal{F}_e$

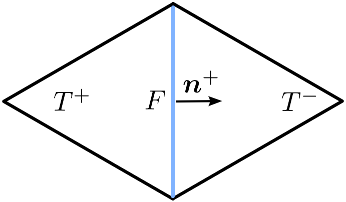
Redistribute integrals over cell boundaries into integrals over facets $\mathcal{F}$ as follows:

$$\sum_{T\in\mathcal{T}}\int_{\partial T}\operatorname{grad} u\cdot n\, v\,\mathrm{d}s = \sum_{e\in\mathcal{F}_i}\int_e (\operatorname{grad} u^+\cdot n^+\, v^+ + \operatorname{grad} u^-\cdot n^-\, v^-)\,\mathrm{d}s \\ + \sum_{e\in\mathcal{F}_e}\int_e \operatorname{grad} u\cdot n\, v\,\mathrm{d}s$$

Let us say that $n^+ = n$ (then $n^- = -n$), and rewrite

$$\sum_{T\in\mathcal{T}}\int_{\partial T}\operatorname{grad} u\cdot n\, v\,\mathrm{d}s = \sum_{e\in\mathcal{F}_i}\int_e (\operatorname{grad} u^+\cdot n\, v^+ - \operatorname{grad} u^-\cdot n\, v^-)\,\mathrm{d}s \\ + \sum_{e\in\mathcal{F}_e}\int_e \operatorname{grad} u\cdot n\, v\,\mathrm{d}s$$

Discontinuous Galerkin: notation



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Jump identity (Exercise for the reader!)

$$[\![Bv]\!] = [\![B]\!]\langle v \rangle + \langle B \rangle [\![v]\!]$$

Deriving a DG formulation (iii)

Now, let's introduce our shorthand notation for the jump:

$$\sum_{T \in \mathcal{T}} \int_{\partial T} \operatorname{grad} u \cdot n \, v \, \mathrm{d}s = \sum_{e \in \mathcal{F}_i} \int_e [\![\operatorname{grad} u v]\!] \, \mathrm{d}s + \sum_{e \in \mathcal{F}_e} \int_e \operatorname{grad} u \cdot n \, v \, \mathrm{d}s$$

Use the jump identity to expand the first term

$$\begin{aligned} \sum_{T \in \mathcal{T}} \int_{\partial T} \operatorname{grad} u \cdot n \, v \, \mathrm{d}s = & \sum_{e \in \mathcal{F}_i} \int_e [\![\operatorname{grad} u]\!] \langle v \rangle + \langle \operatorname{grad} u \rangle [\![v]\!] \, \mathrm{d}s \\ & + \sum_{e \in \mathcal{F}_e} \int_e \operatorname{grad} u \cdot n \, v \, \mathrm{d}s \end{aligned}$$

Deriving a DG formulation (iv)

We want to weakly enforce

- Continuity of the flux: $[\![\operatorname{grad} u]\!] = 0$ over all facets
  (Solution: Let the corresponding term vanish)
- Continuity of the solution $[\![u]\!] = 0$ over all facets
  (Solution: Add a corresponding term)
- Stability
  (Solution: add:

$$S(u, v) = \sum_{e \in \mathcal{F}} \int_e \frac{\beta}{h} [\![u]\!] \cdot [\![v]\!] \, \mathrm{d}s$$

for some stabilization parameter $\beta > 0$ and mesh size h.)

A symmetric interior penalty (SIP/DG) formulation for Poisson's equation

Find $u \in V_h = DG_k(\mathcal{T})$ such that

$$\sum_{T \in \mathcal{T}} \int_T \operatorname{grad} u \cdot \operatorname{grad} v \, \mathrm{d}x$$

$$+ \sum_{e \in \mathcal{F}_i} \int_e -\langle \operatorname{grad} u \rangle [\![v]\!] - \langle \operatorname{grad} v \rangle [\![u]\!] + \tfrac{\alpha}{h} [\![u]\!] \cdot [\![v]\!] \, \mathrm{d}s$$

$$+ \sum_{e \in \mathcal{F}_e} \int_e -\operatorname{grad} u \cdot n \, v - \operatorname{grad} v \cdot n \, u + \tfrac{\alpha}{h} u \, v \, \mathrm{d}s = \int_\Omega f v \, \mathrm{d}x$$

for all $v \in DG_k(\mathcal{T})$.

Useful FEniCS tools for DG (I)

Access facet normals and local mesh size:

```
mesh = UnitSquareMesh(8, 8)
n = FacetNormal(mesh)
h = mesh.hmin()
```

Restrictions:

```
V = FunctionSpace(mesh, "DG", 0)
f = Function(V)
f('+')
grad(f)('+')
```

Useful FEniCS tools for DG (II)

Average and jump:

```
# Define it yourself
h_avg = (h('+') + h('-'))/2

# Or use built-in expression(s)
avg(h)

# This is v^+ - v^-
jump(v)

# This is (v^+ - v^-) n
jump(v, n)
```

Useful FEniCS tools for DG (III)

Integration over sum of all *interior* facets: `dS`:

```
alpha = Constant(0.1)
u = TrialFunction(V)
v = TestFunction(V)
S = alpha/h_avg*dot(jump(v, n), jump(u, n))*dS
```

Integration over sum of all *exterior* facets: `ds`:

```
s = alpha/h*u*v*ds
```

FEniCS programming exercise

We consider our favorite Poisson problem on $\Omega = [0, 1] \times [0, 1]$ with $f = 1.0$.

Solve this PDE numerically by using the SIP/DG method. Try using different values for the stabilization parameter. How does the parameter affect the result?

Compare the solution with the solution obtained using the method of Lecture 02.