FEniCS Course

Lecture 9: Incompressible Navier–Stokes

Contributors
Anders Logg
Marie E. Rognes

Definitions

- Let Ω be a domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with coordinates x
- The boundary is $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$
- $u : \Omega \to \mathbb{R}^d$ is the unknown fluid velocity
- $p : \Omega \to \mathbb{R}$ is the unknown fluid pressure
- $\nabla u = \{\frac{\partial u_j}{\partial x_i}\}_{i,j=1}^d$ NB!
- $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the strain rate tensor
- $\sigma(u, p) = 2\mu\varepsilon(u) - pI$ is the Cauchy stress tensor
- $\rho \in \mathbb{R}$ is a given fluid density
- f is a given body force per unit volume
- g_{D} is a given boundary velocity
- t_{N} is a given boundary traction
- u_0 is a given initial velocity

The incompressible Navier–Stokes equations

Constitutive equations

$$\begin{aligned} \rho(\dot{u} + u \cdot \nabla u) - \nabla \cdot \sigma(u, p) &= f \quad \text{in } \Omega \times (0, T] \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0, T] \end{aligned}$$

Boundary conditions

$$\begin{aligned} u &= g_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \times (0, T] \\ \sigma \cdot n &= t_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}} \times (0, T] \end{aligned}$$

Initial condition

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega$$

Mixed variational formulation of Navier–Stokes

Multiply the momentum equation by a test function v and integrate by parts:

$$\int_\Omega \rho(\dot{u}+u\cdot\nabla u)\cdot v\,\mathrm{d}x+\int_\Omega \sigma(u,p):\varepsilon(v)\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x+\int_{\Gamma_\mathrm{N}} t_\mathrm{N}\cdot v\,\mathrm{d}s$$

Short-hand notation: $\langle\cdot,\cdot\rangle$ is L^2-inner product

$$\langle \rho\dot{u}, v\rangle + \langle \rho u\cdot\nabla u, v\rangle + \langle \sigma(u,p), \varepsilon(v)\rangle = \langle f, v\rangle + \langle t_\mathrm{N}, v\rangle_{\Gamma_\mathrm{N}}$$

Multiply the continuity equation by a test function q and sum up: find $(u,p)\in V$ such that

$$\langle \rho\dot{u}, v\rangle + \langle \rho u\cdot\nabla u, v\rangle + \langle \sigma(u,p), \varepsilon(v)\rangle + \langle \nabla\cdot u, q\rangle = \langle f, v\rangle + \langle t_\mathrm{N}, v\rangle_{\Gamma_\mathrm{N}}$$

for all $(v,q)\in\hat{V}$

Discrete mixed variational form of Navier–Stokes

Time-discretization leads to a *saddle-point* problem on each time step:

$$\begin{bmatrix} M + \Delta t A + \Delta t N(U) & \Delta t B \\ \Delta t B^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- Efficient solution of the saddle-point problem relies on the efficiency of special-purpose preconditioners (Uzawa iteration, Schur complement preconditioners, ...)
- We will use another approach (simpler and often more efficient)

Splitting scheme for Navier-Stokes: Core idea

- Solving the full coupled system for the velocity and the pressure simultaneously is computationally expensive.
- To reduce computational cost, iterative *splitting schemes* are an attractive alternative.
- Splitting schemes are typically based on solving for the velocity and the pressure separately.
- We will consider a splitting scheme solving three different (smaller!) systems at each time step n:
  1. Compute the *tentative velocity*
  2. Compute the *pressure*
  3. Compute the *corrected velocity*
- Next slides show how the scheme is derived – time to pay close attention!

A splitting scheme for Navier–Stokes (Part 1/3)

Recall momentum equation:

$$\rho(\dot{u} + u \cdot \nabla u) - \nabla \cdot \sigma(u, p) = f$$

Consider a time step (t_{n-1}, t_n) of length $k_n = t_n - t_{n-1}$, and introduce

$$\dot{u}(t^n) \approx D_t u^n \equiv (u^n - u^{n-1})/k_n$$

Assume u^{n-1} is given, want to compute u^n.

A Crank-Nicolson approximation with explicit convection for the time-discretization gives

$$\rho D_t u^n + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-1/2}) = f^{n-1/2}$$

Define the tentative velocity $u^\star$ using the approximation

$$\rho D_t u^\star + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-3/2}) = f^{n-1/2} \quad (1)$$

A splitting scheme for Navier-Stokes (Part 2/3)

Subtract the equation for the tentative velocity from the equation for the corrected velocity u^n:

$$\rho(D_t u^n - D_t u^\star) - \nabla \cdot \sigma(0, p^{n-1/2} - p^{n-3/2}) = 0$$

Definition of D_t gives:

$$\rho(D_t u^n - D_t u^\star) = \rho\left(\frac{u^n - u^{n-1}}{k_n} - \frac{u^\star - u^{n-1}}{k_n}\right) = \frac{\rho}{k_n}(u^n - u^\star)$$

Definition of $\sigma(u,p) = 2\mu\varepsilon(u) - pI$ gives:

$$\begin{aligned} -\nabla \cdot \sigma(0, p^{n-1/2} - p^{n-3/2}) &= \nabla \cdot (p^{n-1/2} - p^{n-3/2})I \\ &= \nabla(p^{n-1/2} - p^{n-3/2}) \end{aligned}$$

Multiplying by k_n and keeping only u^n on the left hand side give

$$\rho u^n = \rho u^\star - k_n \nabla(p^{n-1/2} - p^{n-3/2}) \tag{2}$$

A splitting scheme for Navier-Stokes (Part 3/3)

Recall (1):

$$\rho u^n = \rho u^\star - k_n \nabla (p^{n-1/2} - p^{n-3/2})$$

Assuming $u^\star$ and $p^{n-3/2}$ given, need $p^{n-1/2}$!

Taking the divergence of (1) gives

$$\rho \nabla \cdot u^n = \nabla \cdot \Big(\rho u^\star - k_n \nabla (p^{n-1/2} - p^{n-3/2}) \Big)$$

We want $\nabla \cdot u^n = 0$, i.e

$$0 = \nabla \cdot \Big(\rho u^\star - k_n \nabla (p^{n-1/2} - p^{n-3/2}) \Big)$$

or equivalently (denoting $\Delta p = \nabla \cdot \nabla p$):

$$-k_n \Delta p^{n-1/2} = -k_n \Delta p^{n-3/2} - \rho \nabla \cdot u^\star \tag{3}$$

A splitting scheme for Navier-Stokes (Summary)

For each n, given u^{n-1} and $p^{n-3/2}$,

- **Step 1**: Compute the tentative velocity $u^\star$ from

$$\rho D_t u^\star + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-3/2}) = f^{n-1/2}$$

- **Step 2**: Compute the pressure $p^{n-1/2}$ from

$$-k_n \Delta p^{n-1/2} = -k_n \Delta p^{n-3/2} - \rho \nabla \cdot u^\star$$

- **Step 3**: Compute the corrected velocity u^n from

$$\rho u^n = \rho u^\star - k_n \nabla (p^{n-1/2} - p^{n-3/2})$$

Boundary conditions

We consider boundary conditions of the type

$$\begin{aligned} u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \times (0, T] \\ \sigma \cdot n = t_{\mathrm{N}} &= -\bar{p} n && \text{on } \Gamma_{\mathrm{N}} \times (0, T] \end{aligned}$$

- Velocity boundary conditions ($u = g_D$ on Γ_D) are enforced strongly in the finite element spaces, i.e as Dirichlet boundary conditions.
- Traction boundary conditions ($\sigma \cdot n = t_{\mathrm{N}}$) are enforced weakly in the variational formulation.

In the splitting scheme, auxilliary boundary conditions are required for the pressure Poisson problem:

$$\begin{aligned} p &= \bar{p} && \text{on } \Gamma_{\mathrm{N}} \\ \partial_n \dot{p} &= 0 && \text{on } \Gamma_{\mathrm{D}} \end{aligned}$$

and an auxiliary initial condition for the pressure $p^{-1/2} = p_0$.

Enforcing traction boundary conditions in splitting scheme

Note that

$$\begin{aligned}\sigma(u^{n-1/2}, p^{n-3/2}) &= 2\mu\varepsilon(u^{n-1/2}) - p^{n-3/2}I - p^{n-1/2}I + p^{n-1/2}I \\ &= \sigma(u^{n-1/2}, p^{n-1/2}) + p^{n-1/2} - p^{n-3/2}\end{aligned}$$

If we want to enforce

$$\sigma(u^{n-1/2}, p^{n-1/2}) \cdot n = -\bar{p}n$$

and

$$p^{n-1/2} \cdot n = \bar{p}n$$

Then, we should say

$$\sigma(u^{n-1/2}, p^{n-3/2}) \cdot n = -p^{n-3/2}n$$

Weak formulation of N-S splitting scheme

For $n = 1, 2, \ldots, N$, given u^0 and $p^{-1/2}$:

1. Compute $u^\star$ with $u^\star|_{\Gamma_D} = g_D$ solving

$$\langle \rho D_t^n u^\star, v \rangle + \langle \rho u^{n-1} \cdot \nabla u^{n-1}, v \rangle + \langle \sigma(u^{n-\frac{1}{2}}, p^{n-3/2}), \varepsilon(v) \rangle = \langle f^{n-1/2}, v \rangle - \langle p^{n-3/2} n, v \rangle_{\partial\Omega}$$

   for all v such that $v|_{\Gamma_D} = 0$.

2. Compute $p^{n-1/2}$ with $p^{n-1/2}|_{\Gamma_N} = \bar{p}$

$$k_n \langle \nabla p^{n-1/2}, \nabla q \rangle = k_n \langle \nabla p^{n-3/2}, \nabla q \rangle - \langle \rho \nabla \cdot u^\star, q \rangle$$

   for all q such that $q|_{\Gamma_N} = 0$

3. Compute u^n solving

$$\langle \rho u^n, v \rangle = \langle \rho u^\star, v \rangle - k_n \langle \nabla (p^{n-1/2} - p^{n-3/2}), v \rangle$$

   for all v.

Useful FEniCS tools (I)

Note grad vs. ∇:

```
dot(grad(u), u)
dot(u, nabla_grad(u))
```

Defining operators:

```
def sigma(u, p):
    return 2.0*mu*sym(grad(u))-p*Identity(len(u))
```

The facet normal n:

```
n = FacetNormal(mesh)
```

Useful FEniCS tools (II)

Assembling matrices and vectors:

```
A = assemble(a)
b = assemble(L)
```

Solving linear systems:

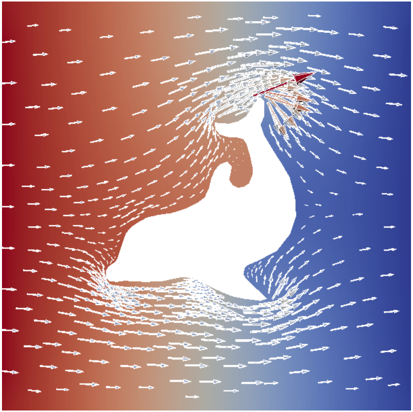
```
solve(A, x, b)
solve(A, x, b, "gmres", "ilu")
solve(A, x, b, "cg", "amg")
```

Extracting left- and right-hand sides:

```
F = <complicated expression>
a = lhs(F)
L = rhs(F)
```

The FEniCS challenge!

Solve the incompressible Navier–Stokes equations for the flow of water around a dolphin. The water is initially at rest and the flow is driven by a pressure gradient.



The FEniCS challenge!

- Use the mesh `dolfin_channel.xml.gz`, and finite element spaces $V_h = \mathcal{P}_2^2$ and $Q_h = \mathcal{P}_1$
- Compute the solution on the time interval $[0, 0.1]$ with time steps of size $k = 0.0005$
- Set $\bar{p} = 1$ kPa at the inflow (left side) and $\bar{p} = 0$ at the outflow (right side)
- Set $g_{\mathrm{D}} = (0, 0)$ on the remaining boundary
- Set $f = (0, 0)$
- The density of water is $\rho = 1000$ kg/m^3 and the viscosity is $\mu = 0.001002$ kg/(m $\cdot$ s)
- To check your answer, compute the average velocity in the x-direction:

$$\bar{u}_x = \frac{1}{|\Omega|} \int_\Omega u \cdot (1, 0) \, \mathrm{d}x$$

The student(s) who first produce the right answer will be rewarded with an exclusive FEniCS surprise!