

FEniCS Course

Lecture 4: Time-dependent PDEs

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The heat equation

We will solve the simplest extension of the Poisson problem into the time domain, the heat equation:

$$\begin{aligned}\frac{\partial u}{\partial t} - \Delta u &= f \quad \text{in } \Omega \text{ for } t > 0 \\ u &= g \quad \text{on } \partial\Omega \text{ for } t > 0 \\ u &= u^0 \quad \text{in } \Omega \text{ at } t = 0\end{aligned}$$

The solution $u = u(x, t)$, the right-hand side $f = f(x, t)$ and the boundary value $g = g(x, t)$ may vary in space ($x = (x_0, x_1, \dots)$) and time (t). The initial value u^0 is a function of space only.

Time-discretization of the heat equation

We discretize in time using the implicit Euler (dG(0)) method:

$$\frac{\partial u}{\partial t}(t^n) \approx \frac{u^n - u^{n-1}}{\Delta t}, \quad u(t^n) \approx u^n, \quad f^n = f(t^n)$$

Semi-discretization of the heat equation:

$$\frac{u^n - u^{n-1}}{\Delta t} - \Delta u^n = f^n$$

Algorithm

- 1 Start with u^0 and choose a timestep $\Delta t > 0$.
- 2 For $n = 1, 2, \dots$, solve for u^n :

$$u^n - \Delta t \Delta u^n = u^{n-1} + \Delta t f^n$$

Variational problem for the heat equation

Find $u^n \in V^n$ such that

$$a(u^n, v) = L^n(v)$$

for all $v \in \hat{V}$ where

$$a(u, v) = \int_{\Omega} uv + \Delta t \nabla u \cdot \nabla v \, dx$$
$$L^n(v) = \int_{\Omega} u^{n-1}v + \Delta t f^n v \, dx$$

Note that the bilinear form $a(u, v)$ is constant while the linear form L^n depends on n

Detailed time-stepping algorithm for the heat equation

Define the boundary condition

Compute u^0 as the projection of the given initial value

Define the forms a and L

Assemble the matrix A from the bilinear form a

$t \leftarrow \Delta t$

while $t \leq T$ **do**

Assemble the vector b from the linear form L

Apply the boundary condition

Solve the linear system $AU = b$ for U and store in u^1

$t \leftarrow t + \Delta t$

$u^0 \leftarrow u^1$ (get ready for next step)

end while

Test problem

We construct a test problem for which we can easily check the answer. We first define the exact solution by

$$u = 1 + x^2 + \alpha y^2 + \beta t$$

We insert this into the heat equation:

$$f = \dot{u} - \Delta u = \beta - 2 - 2\alpha$$

The initial condition is

$$u^0 = 1 + x^2 + \alpha y^2$$

This technique is called the *method of manufactured solutions*

Handling time-dependent expressions

We define a time-dependent expression for the boundary value:

```
alpha = 3; beta = 1.2
t = 0.0
g = Expression("1 + x[0]*x[0] + \
               alpha*x[1]*x[1] + beta*t",
               alpha=alpha, beta=beta, t=t,
               degree=2)
```

Then, we must explicitly update t and g:

```
t = 1.0
g.t = t
```

An alternative (robust) approach is to define t as a Constant:

```
t = Constant(0.0)
g = Expression("...", ..., t=t, ...)
t.assign(1.0)
# No need to update g itself
```

Projection and interpolation

We need to project the initial value into V_h :

$$u_0 = \text{project}(g, v)$$

We can also interpolate the initial value into V_h :

$$u_0 = \text{interpolate}(g, v)$$

A closer look at solve

For linear problems, this code

```
solve(a == L, u, bcs)
```

is equivalent to this

```
# Assembling a bilinear form yields a matrix  
A = assemble(a)  
# Assembling a linear form yields a vector  
b = assemble(L)  
  
# Applying boundary condition info to system  
for bc in bcs:  
    bc.apply(A, b)  
  
# Solve Ax = b  
solve(A, u.vector(), b)
```

Implementing the variational problem

```
# Decide on a time step
dt = 0.3

# Create Functions for previous and current sol.s
u0 = project(g, V)
u1 = Function(V)

# Define the variational formulation
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(beta - 2 - 2*alpha)
a = u*v*dx + dt*inner(grad(u), grad(v))*dx
L = u0*v*dx + dt*f*v*dx

# Define the boundary condition
bc = DirichletBC(V, g, "on_boundary")

# Assemble only once, before time-stepping
A = assemble(a)
```

Implementing the time-stepping loop

```
T = 2          # Set end time
t.assign(dt)  # Solve on [0, dt] first

while t <= T:
    b = assemble(L) # Assemble the rhs vector
    bc.apply(A, b)  # Apply boundary conditions

    # Solve linear system
    solve(A, u1.vector(), b)

    # Update time and previous solution
    t1 = float(t + dt)
    t.assign(t1)          # t := t1 + dt
    u0.assign(u1)        # u0 := u1
```

FEniCS programming exercise: heat equation

Consider the heat equation problem:

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega = [0, 1]^2 \text{ for } t > 0$$

$$u(x, t) = g(x, t) \quad \text{for } x \in \partial\Omega \text{ for } t > 0$$

$$u(x, 0) = g(x, 0) \quad \text{for } x \in \Omega$$

with

$$f = \beta - 2 - 2\alpha$$

$$g(x, t) = 1 + x_0^2 + \alpha x_1^2 + \beta t \quad (x = (x_0, x_1))$$

Ex. 1 Compute an approximate solution at $T = 1.8$

Ex. 2 Compare the approximate solution to the exact solution at $T = 1.8$. How large is the error (in the eyenorm and in the $L^2(\Omega)$ norm)?

Ex. 3 Compute an approximate solution with the same set-up but on $\Omega = [0, 1]^3 \subset \mathbb{R}^3$.