FEniCS Course

Lecture 7: Dynamic hyperelasticity

Contributors
Anders Logg

Dynamic hyperelasticity

$$\begin{aligned}
\rho\ddot{u} - \operatorname{div} P = B \quad & \text{in } \Omega \times (0,T] \\
u = g \quad & \text{on } \Gamma_{\mathrm{D}} \times (0,T] \\
P \cdot n = T \quad & \text{on } \Gamma_{\mathrm{N}} \times (0,T] \\
u(\cdot,0) = u_0 \quad & \text{in } \Omega \\
\dot{u}(\cdot,0) = u_1 \quad & \text{in } \Omega
\end{aligned}$$

- u is the displacement
- ρ is the (reference) density
- $P = P(u)$ is the first Piola–Kirchoff stress tensor
- B is a given body force per unit volume
- g is a given boundary displacement
- T is a given boundary traction
- u_0 and u_1 are given initial displacement and velocity

Variational problem

Rewrite as a first-order system by introducing $p = \dot{u}$:

$$\rho \dot{p} - \operatorname{div} P = B$$
$$\dot{u} - p = 0$$

Multiply by test functions v and q and sum up:

$$\int_{t_{n-1}}^{t_n} \int_\Omega (\rho \dot{p} - \operatorname{div} P) \cdot v \,\mathrm{d}x \,\mathrm{d}t + \int_{t_{n-1}}^{t_n} \int_\Omega (\dot{u} - p) \cdot q \,\mathrm{d}x \,\mathrm{d}t = \int_{t_{n-1}}^{t_n} \int_\Omega B \cdot v \,\mathrm{d}x$$

Integrate by parts and use $v = 0$ on Γ_{D} and $P \cdot n = T$ on Γ_{N}:

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} \int_\Omega \rho \dot{p} \cdot v \,\mathrm{d}x \,\mathrm{d}t + \int_{t_{n-1}}^{t_n} \int_\Omega P : \operatorname{grad} v \,\mathrm{d}x \,\mathrm{d}t \\
+ \int_{t_{n-1}}^{t_n} \int_\Omega \dot{u} \cdot q \,\mathrm{d}x \,\mathrm{d}t - \int_{t_{n-1}}^{t_n} \int_\Omega p \cdot q \,\mathrm{d}x \,\mathrm{d}t \\
= \int_{t_{n-1}}^{t_n} \int_\Omega B \cdot v \,\mathrm{d}x \,\mathrm{d}t + \int_{t_{n-1}}^{t_n} \int_{\Gamma_{\mathrm{N}}} T \cdot v \,\mathrm{d}s \,\mathrm{d}t
\end{aligned}$$

Time discretization

Let the trial functions u, p be continuous and piecewise linear in time, and let the test functions v, q be piecewise constant:

$$\int_{t_{n-1}}^{t_n} \int_\Omega \rho \dot{p} \cdot v \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega \rho (p(\cdot, t_n) - p(\cdot, t_{n-1})) \cdot v \, \mathrm{d}x$$

$$\int_{t_{n-1}}^{t_n} \int_\Omega \dot{u} \cdot q \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega (u(\cdot, t_n) - u(\cdot, t_{n-1})) \cdot q \, \mathrm{d}x$$

$$\int_{t_{n-1}}^{t_n} \int_\Omega p \cdot q \, \mathrm{d}x \, \mathrm{d}t = k_n \int_\Omega p(\cdot, t_{n-1/2}) \cdot q \, \mathrm{d}x$$

where $k_n = t_n - t_{n-1}$ and $p(\cdot, t_{n-1/2}) = p(\cdot, t_n - k_n/2)$
Approximate other integrals by midpoint quadrature:

$$\int_{t_{n-1}}^{t_n} \int_\Omega P : \operatorname{grad} v \, \mathrm{d}x \, \mathrm{d}t \approx k_n \int_\Omega P(u(\cdot, t_{n-1/2})) : \operatorname{grad} v \, \mathrm{d}x$$

This is the cG(1) or *Crank–Nicolson* method

Discrete problem

Find $(u^n, p^n) \in V_h$ such that

$$\begin{aligned}\int_\Omega \rho(p^n - p^{n-1}) \cdot v \, \mathrm{d}x + k_n \int_\Omega P(u^{n-1/2}) : \operatorname{grad} v \, \mathrm{d}x \\ + \int_\Omega (u^n - u^{n-1}) \cdot q \, \mathrm{d}x - k_n \int_\Omega p^{n-1/2} \cdot q \, \mathrm{d}x \\ = k_n \int_\Omega B^{n-1/2} \cdot v \, \mathrm{d}x + k_n \int_{\Gamma_\mathrm{N}} T^{n-1/2} \cdot v \, \mathrm{d}s\end{aligned}$$

for all $(v, q) \in \hat{V}_h$

Stress–strain relations

- $F = I + \operatorname{grad} u$ is the deformation gradient
- $C = F^\top F$ is the right Cauchy–Green tensor
- $E = \frac{1}{2}(C - I)$ is the Green–Lagrange strain tensor
- $W = W(E)$ is the strain energy density
- $S_{ij} = \frac{\partial W}{\partial E_{ij}}$ is the second Piola–Kirchoff stress tensor
- $P = FS$ is the first Piola–Kirchoff stress tensor

St. Venant–Kirchoff strain energy function:

$$W(E) = \frac{\lambda}{2}(\operatorname{tr}(E))^2 + \mu \operatorname{tr}(E^2)$$

Useful FEniCS tools (I)

Defining mixed function spaces:

```
V = VectorFunctionSpace(mesh, "CG", 1)
VV = V*V
```

Defining subfunctions:

```
up = Function(VV)
u, p = split(up)
```

Shortcut:

```
u, p = Functions(VV)
```

Useful FEniCS tools (II)

Time-stepping

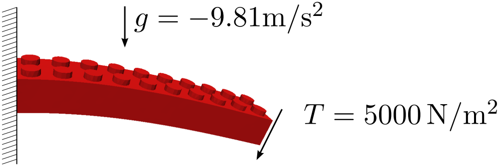
```python
t = dt
while t < T + DOLFIN_EPS:

    # Solve variational problem
    solve(...)

    # Move to next interval
    t += dt
    u0.assign(u1)
    # use up0.assign(up1) for a mixed system
```

The FEniCS challenge!

Compute the deflection of a regular 10 × 2 LEGO brick as function of time. Use the St. Venant–Kirchhoff model and assume that the LEGO brick is made of PVC plastic. The LEGO brick is subject to gravity of size $g = -9.81$ m/s^2 and a downward traction of size 5000 N/m^2 at its end point. At time $t = 0$, the brick is at rest in its undeformed state.



To check your solution, compute the average value of the displacement in the z-direction at time $T = 0.05$. Use a time step of size $k = 0.002$.