FEniCS Course

Lecture 4: Time-dependent PDEs

Contributors
Hans Petter Langtangen
Anders Logg
Marie E. Rognes

The heat equation

We will solve the simplest extension of the Poisson problem into the time domain, the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= f \ \text{ in } \Omega \text{ for } t > 0 \\ u &= g \ \text{ on } \partial\Omega \text{ for } t > 0 \\ u &= u^0 \text{ in } \Omega \text{ at } t = 0 \end{aligned}$$

The solution $u = u(x,t)$, the right-hand side $f = f(x,t)$ and the boundary value $g = g(x,t)$ may vary in space ($x = (x_0, x_1, ...)$) and time (t). The initial value u^0 is a function of space only.

Time-discretization of the heat equation

We discretize in time using the implicit Euler (dG(0)) method:

$$\frac{\partial u}{\partial t} \approx \frac{u^n - u^{n-1}}{\Delta t}$$

Semi-discretization of the heat equation:

$$\frac{u^n - u^{n-1}}{\Delta t} - \Delta u^n = f^n$$

$$u^n - \Delta t \Delta u^n = u^{n-1} + \Delta t f^n$$

Solve for u^1, u^2, ...

Variational problem for the heat equation

Find $u^n \in V^n$ such that

$$a(u^n, v) = L^n(v)$$

for all $v \in \hat{V}$ where

$$a(u, v) = \int_\Omega uv + \Delta t \nabla u \cdot \nabla v \, \mathrm{d}x$$
$$L^n(v) = \int_\Omega u^{n-1} v + \Delta t f^n v \, \mathrm{d}x$$

Note that the bilinear form $a(u, v)$ is constant while the linear form L^n depends on n

Pseudocode for a naive implementation of the heat equation

```
from dolfin import *

# Mesh and function space
mesh = UnitCube(8, 8, 8)
V = FunctionSpace(mesh, "CG", 1)

# Time variables
dt = 0.01; k = Constant(dt); t = dt; T = 1.0

# Previous and current solution
u0 = Function(V); u0.vector()[:] = 1.0
u1 = Function(V)

# Variational problem at each time
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("t", t=t)
a = u*v*dx + k*inner(grad(u), grad(v))*dx
L = u0*v*dx + k*f*v*dx
bc = DirichletBC(V, 0.0, "near(x[0], 0.0)")

while (t <= T):
    # Solve
    f.t = t
    solve(a == L, u1, bc)

    # Update
    u0.assign(u1)
    t += dt
    plot(u1)
```

Time-stepping algorithm

Define the boundary condition
Compute u^0 *as the projection of the given initial value*
Define the forms a *and* L
Assemble the matrix A *from the bilinear form* a
$t \leftarrow \Delta t$
while $t \leqslant T$ **do**
  Assemble the vector b *from the linear form* L
  Apply the boundary condition
  Solve the linear system $AU = b$ *for* U *and store in* u^1
  $t \leftarrow t + \Delta t$
  $u^0 \leftarrow u^1$ (get ready for next step)
end while

Test problem

We construct a test problem for which we can easily check the answer. We first define the exact solution by

$$u = 1 + x^2 + \alpha y^2 + \beta t$$

We insert this into the heat equation:

$$f = \dot{u} - \Delta u = \beta - 2 - 2\alpha$$

The initial condition is

$$u^0 = 1 + x^2 + \alpha y^2$$

This technique is called the *method of manufactured solutions*

Handling time-dependent expressions

We need to define a time-dependent expression for the boundary value:

```
alpha = 3
beta = 1.2

g = Expression("1 + x[0]*x[0] + \
               alpha*x[1]*x[1] + beta*t",
               alpha=alpha, beta=beta, t=0)
```

Updating parameter values:

```
g.t = t
```

Projection and interpolation

We need to project the initial value into V_h:

```
u0 = project(g, V)
```

We can also interpolate the initial value into V_h:

```
u0 = interpolate(g, V)
```

A closer look at `solve`

For linear problems, this code

```
solve(a == L, u, bcs)
```

is equivalent to this

```
# Assembling a bilinear form yields a matrix
A = assemble(a)
# Assembling a linear form yields a vector
b = assemble(L)

# Applying boundary condition info to system
for bc in bcs:
    bc.apply(A, b)

# Solve Ax = b
solve(A, u.vector(), b)
```

Implementing the variational problem

```
dt = 0.3

u0 = project(g, V)
u1 = Function(V)

u = TrialFunction(V)
v = TestFunction(V)
f = Constant(beta - 2 - 2*alpha)

a = u*v*dx + dt*inner(grad(u), grad(v))*dx
L = u0*v*dx + dt*f*v*dx

bc = DirichletBC(V, g, "on_boundary")

# assemble only once, before time-stepping
A = assemble(a)
```

Implementing the time-stepping loop

```
T = 2
t = dt

while t <= T:
    b = assemble(L)
    g.t = t
    bc.apply(A, b)
    solve(A, u1.vector(), b)

    t += dt
    u0.assign(u1)
```

Programming exercise

- Write a program to solve the heat equation
- Write your program in a file named `heat.py`
- Run your program using

```
python heat.py
```

- A complete program suggestion is available[1] as

```
transient/diffusion/d1_d2D.py
```

[1] `http://fenicsproject.org/pub/book/tutorial/`