Geilo Winter School 2012

Lecture 9: Navier–Stokes in FEniCS

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Course outline

Sunday

L1 Introduction to FEM

Monday

L2 Fundamentals of continuum mechanics (I)

L3 Fundamentals of continuum mechanics (II)

L4 Introduction to FEniCS

Tuesday

L5 Solid mechanics

L6 Static hyperelasticity in FEniCS

L7 Dynamic hyperelasticity in FEniCS

Wednesday

L8 Fluid mechanics

L9 Navier–Stokes in FEniCS

The incompressible Navier–Stokes equations

$$
\begin{aligned}
\rho(\dot{u} + u \cdot \nabla u) - \nabla \cdot \sigma(u, p) &= f && \text{in } \Omega \times (0, T] \\
\nabla \cdot u &= 0 && \text{in } \Omega \times (0, T] \\
u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \times (0, T] \\
\sigma \cdot n &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} \times (0, T] \\
u(\cdot, 0) &= u_0 && \text{in } \Omega
\end{aligned}
$$

- u is the fluid velocity and p is the pressure
- ρ is the fluid density
- $\sigma(u, p) = 2\mu\epsilon(u) - pI$ is the Cauchy stress tensor
- $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ is the symmetric gradient
- f is a given body force per unit volume
- g_{D} is a given boundary displacement
- g_{N} is a given boundary traction
- u_0 is a given initial velocity

Variational problem

Multiply the momentum equation by a test function v and integrate by parts:

$$\int_\Omega \rho(\dot{u}+u\cdot\nabla u)\cdot v \,\mathrm{d}x + \int_\Omega \sigma(u,p) : \epsilon(u) \,\mathrm{d}x = \int_\Omega f\cdot v \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g_\mathrm{N}\cdot v \,\mathrm{d}s$$

Short-hand notation:

$$\langle \rho\dot{u}, v\rangle + \langle \rho u\cdot\nabla u, v\rangle + \langle \sigma(u,p), \epsilon(v)\rangle = \langle f, v\rangle + \langle g_\mathrm{N}, v\rangle_{\Gamma_\mathrm{N}}$$

Multiply the continuity equation by a test function q and sum up: find $(u,p) \in V$ such that

$$\langle \rho\dot{u}, v\rangle + \langle \rho u\cdot\nabla u, v\rangle + \langle \sigma(u,p), \epsilon(v)\rangle + \langle \nabla\cdot u, q\rangle = \langle f, v\rangle + \langle g_\mathrm{N}, v\rangle_{\Gamma_\mathrm{N}}$$

for all $(v,q) \in \hat{V}$

Discrete variational problem

Time-discretization leads to a *saddle-point* problem on each time step:

$$\begin{bmatrix} M + \Delta t A + \Delta t N(U) & \Delta t B \\ \Delta t B^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- Efficient solution of the saddle-point problem relies on the efficiency of special-purpose preconditioners (Uzawa iteration, Schur complement preconditioners, ...)
- We will use another approach (simpler and often more efficient)

A splitting method

cG(1) / Crank-Nicolson approximation with explicit convection:

$$\rho D_t u^n + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-1/2}) = f^{n-1/2}$$

Compute the *tentative velocity* $u^\star$ using the approximation

$$\rho D_t u^\star + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, {\color{red}p^{n-3/2}}) = f^{n-1/2}$$

Subtract:

$$\rho(D_t u^n - D_t u^\star) - \nabla \cdot \sigma(u^{n-1/2}, p^{n-1/2} - p^{n-3/2}) = 0$$

Expand and rearrange:

$$\rho(u^n - u^\star) + k_n \nabla(p^{n-1/2} - p^{n-3/2}) = 0$$

$D_t u = (u^n - u^{n-1})/k_n$ and $k_n = t_n - t_{n-1}$

A splitting method (contd.)

We have found that:

$$\rho(u^n - u^\star) + k_n \nabla(p^{n-1/2} - p^{n-3/2}) = 0$$

It follows that

$$\rho u^n = \rho u^\star - k_n \nabla(p^{n-1/2} - p^{n-3/2}) \tag{1}$$

Take the divergence and set $\nabla \cdot u^n = 0$:

$$-k_n \Delta p^{n-1/2} = -k_n \Delta p^{n-3/2} - \rho \nabla \cdot u^\star \tag{2}$$

- Compute $p^{n-1/2}$ by solving the Poisson problem (2)
- Compute u^n by solving the projection problem (1)
- To consider: *What about the boundary conditions for the Poisson problem (2)?*

Boundary conditions

- For outflow boundary conditions, corresponding to so-called "do-nothing" boundary conditions for the Laplacian formulation, we take $\partial_n u = 0$:

$$\begin{aligned}\sigma(u,p)\cdot n &= (2\mu\epsilon(u) - pI)\cdot n = \mu\nabla u\cdot n + \mu(\nabla)^\top\cdot n - pn \\ &= \mu(\nabla u)^\top\cdot n - pn \approx \mu(\nabla u^{n-1/2})^\top\cdot n - p^{n-3/2}n\end{aligned}$$

- Boundary conditions for the pressure Poisson problem:

$$\partial_n \dot{p} = 0$$

on the pressure Neumann boundary

Incremental pressure correction scheme (IPCS)

1. Compute the tentative velocity $u^\star$ by

$$\langle \rho D_t^n u^\star, v\rangle + \langle \rho u^{n-1} \cdot \nabla u^{n-1}, v\rangle + \langle \sigma(u^{n-\frac{1}{2}}, p^{n-3/2}), \epsilon(v)\rangle$$
$$- \langle \mu n \cdot (\nabla u^{n-\frac{1}{2}})^\top, v\rangle_{\partial\Omega} + \langle p^{n-3/2} n, v\rangle_{\partial\Omega} = \langle f^{n-1/2}, v\rangle$$

2. Compute the corrected pressure $p^{n-1/2}$ by

$$k_n \langle \nabla p^{n-1/2}, \nabla q\rangle = k_n \langle \nabla p^{n-3/2}, \nabla q\rangle - \langle \rho \nabla \cdot u^\star, q\rangle$$

3. Compute the corrected velocity u^n by

$$\langle \rho u^n, v\rangle = \langle \rho u^\star, v\rangle - k_n \langle \nabla (p^{n-1/2} - p^{n-3/2}), v\rangle$$

Useful FEniCS tools (I)

Note grad vs. ∇:

```
dot(grad(u), u)
dot(u, nabla_grad(u))
```

Defining operators:

```
def sigma(u, p):
    return 2.0*mu*sym(grad(u)) - p*Identity(2)
```

The facet normal n:

```
n = FacetNormal(mesh)
```

Useful FEniCS tools (II)

Assembling matrices and vectors:

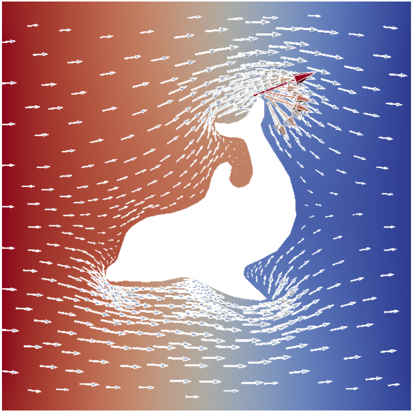
```
A = assemble(a)
b = assemble(L)
```

Solving linear systems:

```
solve(A, x, b)
solve(A, x, b, "gmres", "ilu")
solve(A, x, b, "cg", "amg")
```

The FEniCS challenge!

Solve the incompressible Navier–Stokes equations for the flow of water around a dolphin. The water is initially at rest and the flow is driven by a pressure gradient.



The FEniCS challenge!

- Compute the solution on the time interval $[0, 0.1]$ with time steps of size $k = 0.0005$
- Set $p = 1$ kPa at the inflow and $p = 0$ at the outflow
- The density of water is $\rho = 1000\ \mathrm{kg/m^3}$ and the viscosity is $\mu = 0.001002\ \mathrm{kg/(m \cdot s)}$
- To check your answer, compute the average velocity in the x-direction.

The student(s) who first produce the right anwswer will be rewarded with an exclusive FEniCS coffee mug!