

Automated Solution of Differential Equations with Application to Fluid–Structure Interaction on Cut Meshes

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2012-11-26

* Credits: <http://fenicsproject.org/about/team.html>

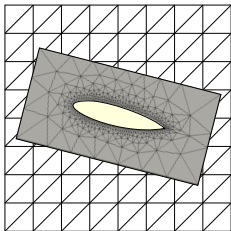


Topics

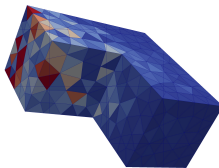
The FEniCS Project



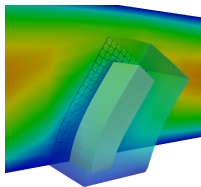
Cut FEM



Adaptivity



FSI



What is FEniCS?

FEniCS is an automated programming environment for differential equations

- C++/Python library
- Initiated 2003 in Chicago
- 1000–2000 monthly downloads
- Part of Debian and Ubuntu
- Licensed under the GNU LGPL



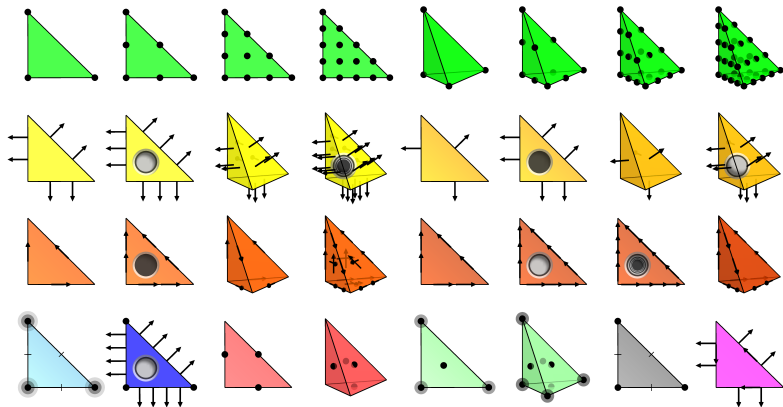
<http://fenicsproject.org/>

Collaborators

*Simula Research Laboratory, University of Cambridge,
University of Chicago, Texas Tech University, University of
Texas at Austin, KTH Royal Institute of Technology, ...*

FEniCS is automated FEM

- Automated generation of basis functions
- Automated evaluation of variational forms
- Automated finite element assembly
- Automated adaptive error control



FEniCS is automated scientific computing

Input

- $A(u) = f$
- $\epsilon > 0$

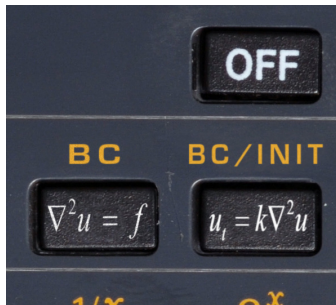
Output

- Approximate solution:

$$u_h \approx u$$

- Guaranteed accuracy:

$$\|u - u_h\| \leq \epsilon$$



How to use FEniCS?

Installation



Official packages for Debian and Ubuntu



Drag and drop installation on Mac OS X



Binary installer for Windows



Automated installation from source

Hello World in FEniCS: problem formulation

Poisson's equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Finite element formulation

Find $u \in V$ such that

$$\underbrace{\int_{\Omega} \nabla u \cdot \nabla v \, dx}_{a(u,v)} = \underbrace{\int_{\Omega} f v \, dx}_{L(v)} \quad \forall v \in V$$

Hello World in FEniCS: implementation

```
from dolfin import *

mesh = UnitSquare(32, 32)

V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("x[0]*x[1]")

a = dot(grad(u), grad(v))*dx
L = f*v*dx

bc = DirichletBC(V, 0.0, DomainBoundary())

u = Function(V)
solve(a == L, u, bc)
plot(u)
```

Linear elasticity

Differential equation

Differential equation:

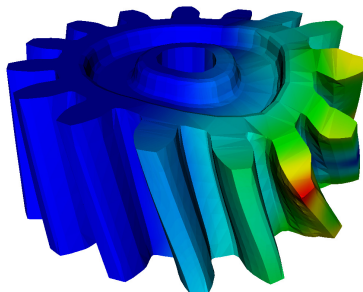
$$-\nabla \cdot \sigma(u) = f$$

where

$$\sigma(v) = 2\mu\epsilon(v) + \lambda\text{tr}\epsilon(v)I$$

$$\epsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^\top)$$

- Displacement $u = u(x)$
- Stress $\sigma = \sigma(x)$



Linear elasticity

Implementation

```
element = VectorElement("Lagrange", "tetrahedron", 1)

v = TestFunction(element)
u = TrialFunction(element)
f = Function(element)

def epsilon(v):
    return 0.5*(grad(v) + grad(v).T)

def sigma(v):
    return 2.0*mu*epsilon(v) + lambda*tr(epsilon(v))*I

a = inner(sigma(u), epsilon(v))*dx
L = dot(f, v)*dx
```

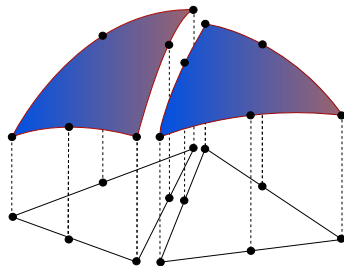
Poisson's equation with DG elements

Differential equation

Differential equation:

$$-\Delta u = f$$

- $u \in L^2$
- u discontinuous across element boundaries



Poisson's equation with DG elements

Variational formulation (interior penalty method)

Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ &+ \sum_S \int_S -\langle \nabla u \rangle \cdot \llbracket v \rrbracket_n - \llbracket u \rrbracket_n \cdot \langle \nabla v \rangle + (\alpha/h) \llbracket u \rrbracket_n \cdot \llbracket v \rrbracket_n \, dS \\ &+ \int_{\partial\Omega} -\nabla u \cdot \llbracket v \rrbracket_n - \llbracket u \rrbracket_n \cdot \nabla v + (\gamma/h) uv \, ds \\ L(v) &= \int_{\Omega} f v \, dx + \int_{\partial\Omega} g v \, ds \end{aligned}$$

Poisson's equation with DG elements

Implementation

```
V = FunctionSpace(mesh, "DG", 1)

u = TrialFunction(V)
v = TestFunction(V)

f = Expression(...)
g = Expression(...)
n = FacetNormal(mesh)
h = CellSize(mesh)

a = dot(grad(u), grad(v))*dx
  - dot(avg(grad(u)), jump(v, n))*dS
  - dot(jump(u, n), avg(grad(v)))*dS
  + alpha/avg(h)*dot(jump(u, n), jump(v, n))*dS
  - dot(grad(u), jump(v, n))*ds
  - dot(jump(u, n), grad(v))*ds
  + gamma/h*u*v*ds
```

Simple prototyping and development in Python

```
# Tentative velocity step (sigma formulation)
U = 0.5*(u0 + u)
F1 = rho*(1/k)*inner(v, u - u0)*dx +
rho*inner(v, grad(u0)*(u0 - w))*dx \
+ inner(epsilon(v), sigma(U, p0))*dx \
+ inner(v, p0*n)*ds - mu*inner(grad(U).T*n, v)*ds \
- inner(v, f)*dx
a1 = lhs(F1)
L1 = rhs(F1)
```

```
class StVenantKirchhoff(MaterialModel):

    def model_info(self):
        self.num_parameters = 2
        self.kinematic_measure = \
            "GreenLagrangeStrain"

    def strain_energy(self, parameters):
        E = self.E
        [mu, lambda] = parameters
        return lambda/2*(tr(E)**2) + mu*tr(E**E)
```

```
class GentThomas(MaterialModel):

    def model_info(self):
        self.num_parameters = 2
        self.kinematic_measure = \
            "CauchyGreenInvariants"

    def strain_energy(self, parameters):
        I1 = self.I1
        I2 = self.I2

        [C1, C2] = parameters
        return C1*(I1 - 3) + C2*ln(I2/3)
```

```
# Time-stepping loop
while True:

    # Fixed point iteration on FSI problem
    for iter in range(maxiter):

        # Solve fluid subproblem
        F.step(dt)

        # Transfer fluid stresses to structure
        Sigma_F = F.compute_fluid_stress(u_F0, u_F1,
                                         p_F0, p_F1,
                                         U_M0, U_M1)

        S.update_fluid_stress(Sigma_F)

        # Solve structure subproblem
        U_S1, P_S1 = S.step(dt)

        # Transfer structure displacement to fluidmesh
        M.update_structure_displacement(U_S1)

        # Solve mesh equation
        M.step(dt)

        # Transfer mesh displacement to fluid
        F.update_mesh_displacement(U_M1, dt)
```

```
# Fluid residual contributions
R_F0 = w*inner(EZ_F - Z_F, Dt_U_F - div(Sigma_F))*dx_F
R_F1 = avg(w)*inner(EZ_F('+') - Z_F('+'),
                   jump(Sigma_F, N_F))*dS_F
R_F2 = w*inner(EZ_F - Z_F, dot(Sigma_F, N_F))*ds
R_F3 = w*inner(EY_F - Y_F,
               div(J(U_M)*dot(inv(F(U_M)), U_F)))*dx_F
```


Simple development of specialized applications



```
# Define Cauchy stress tensor
def sigma(v,w):
    return 2.0*mu*0.5*(grad(v) + grad(v).T) -
w*Identity(v.cell().d)

# Define symmetric gradient
def epsilon(v):
    return 0.5*(grad(v) + grad(v).T)

# Tentative velocity step (sigma formulation)
U = 0.5*(u0 + u)
F1 = rho*(1/k)*inner(v, u - u0)*dx +
rho*inner(v, grad(u0)*(u0 - w))*dx \
+ inner(epsilon(v), sigma(U, p0))*dx \
+ inner(v, p0*n)*ds - mu*inner(grad(U).T*n, v)*ds \
- inner(v, f)*dx
a1 = lhs(F1)
L1 = rhs(F1)

# Pressure correction
a2 = inner(grad(q), k*grad(p))*dx
L2 = inner(grad(q), k*grad(p0))*dx - q*div(u1)*dx

# Velocity correction
a3 = inner(v, u)*dx
L3 = inner(v, u1)*dx + inner(v, k*grad(p0 - p1))*dx
```

- The Navier–Stokes solver is implemented in Python/FEniCS
- FEniCS allows solvers to be implemented in a minimal amount of code
- Simple integration with application specific code and data management

FEniCS under the hood

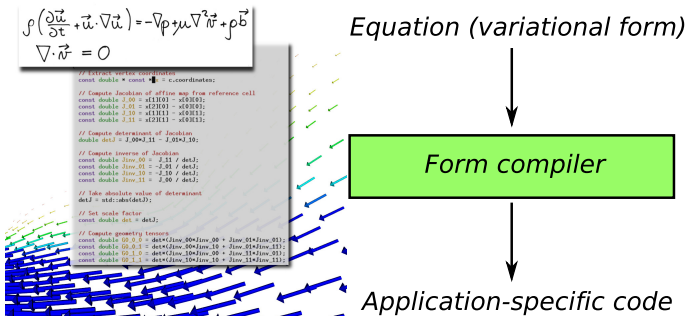
Automatic code generation

Input

Equation (variational problem)

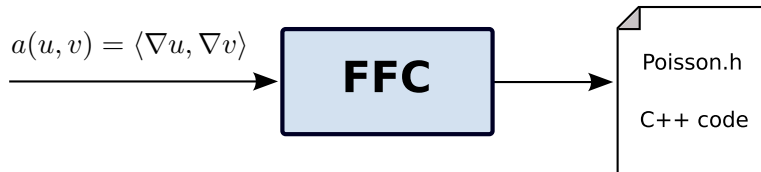
Output

Efficient application-specific code



Code generation framework

- UFL - Unified Form Language
- UFC - Unified Form-assembly Code
- Form compilers: FFC, SyFi



Form compiler interfaces

Command-line

```
>> ffc poisson.ufl
```

Just-in-time

```
V = FunctionSpace(mesh, "CG", 3)
u = TrialFunction(V)
v = TestFunction(V)
A = assemble(dot(grad(u), grad(v))*dx)
```

Code generation system

```
mesh = UnitSquare(32, 32)

V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("x[0]*x[1]")

a = dot(grad(u), grad(v))*dx
L = f*v*dx

bc = DirichletBC(V, 0.0, DomainBoundary())

A = assemble(a)
b = assemble(L)
bc.apply(A, b)

u = Function(V)
solve(A, u.vector(), b)
```

Code generation system

```
mesh = UnitSquare(32, 32)

V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("x[0]*x[1]")

a = dot(grad(u), grad(v))*dx
L = f*v*dx

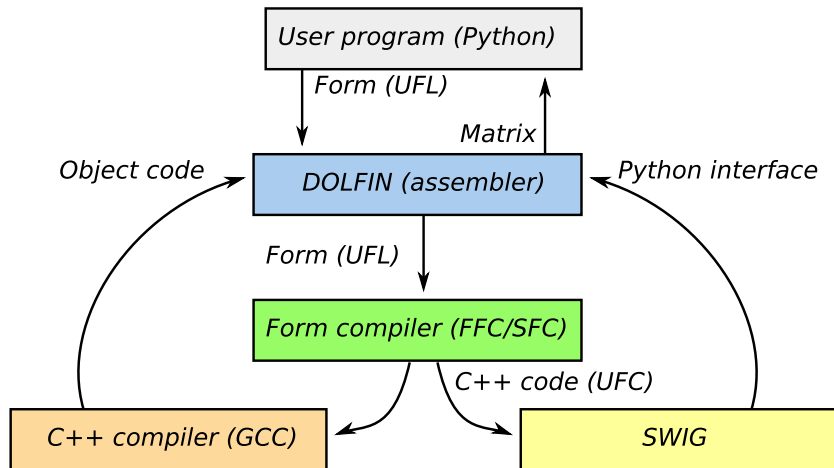
bc = DirichletBC(V, 0.0, DomainBoundary())

A = assemble(a)
b = assemble(L)
bc.apply(A, b)

u = Function(V)
solve(A, u.vector(), b)
```

(Python, C++-SWIG-Python, Python-JIT-C++-GCC-SWIG-Python)

Just-In-Time (JIT) compilation



Automated error control

Automated goal-oriented error control

Input

- Variational problem: Find $u \in V$: $a(u, v) = L(v) \quad \forall v \in V$
- Quantity of interest: $\mathcal{M} : V \rightarrow \mathbb{R}$
- Tolerance: $\epsilon > 0$

Objective

Find $V_h \subset V$ such that $|\mathcal{M}(u) - \mathcal{M}(u_h)| < \epsilon$ where

$$a(u_h, v) = L(v) \quad \forall v \in V_h$$

Automated in FEniCS (for linear and nonlinear PDE)

```
solve(a == L, u, M=M, tol=1e-3)
```

Key steps to automated error control

- Automated linearization
- Automated generation of the dual problem
- Automated integration by parts:

$$r_T(v) = \int_T R_T \cdot v \, dx + \int_{\partial T} R_{\partial T} \cdot v \, ds$$

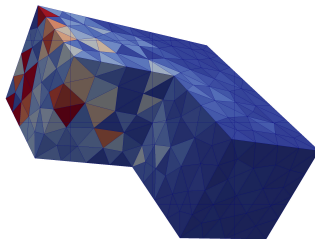
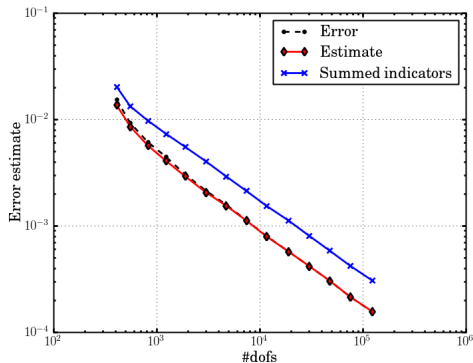
Test against bubble functions to solve for R_T and $R_{\partial T}$

- Automated computation of error indicators:

$$\eta_T = |\langle R_T, \tilde{z}_h - z_h \rangle_T + \langle \llbracket R_{\partial T} \rrbracket, \tilde{z}_h - z_h \rangle_{\partial T}|$$

- Automated mesh refinement
- Dual problem solved on same function space and extrapolated

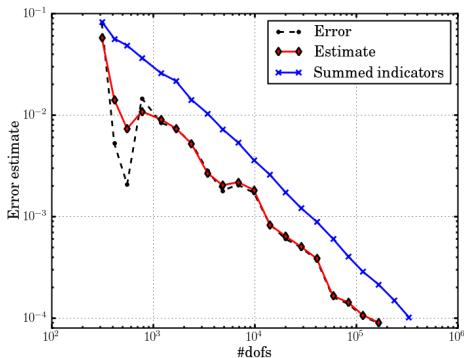
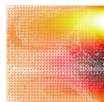
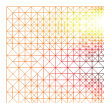
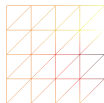
Poisson's equation



$$a(u, v) = \langle \nabla u, \nabla v \rangle$$

$$\mathcal{M}(u) = \int_{\Gamma} u \, ds, \quad \Gamma \subset \partial\Omega$$

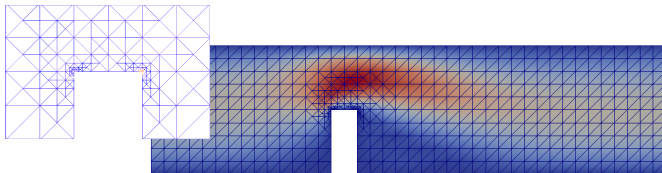
A three-field mixed elasticity formulation



$$a((\sigma, u, \gamma), (\tau, v, \eta)) = \langle A\sigma, \tau \rangle + \langle u, \operatorname{div} \tau \rangle + \langle \operatorname{div} \sigma, v \rangle + \langle \gamma, \tau \rangle + \langle \sigma, \eta \rangle$$

$$\mathcal{M}((\sigma, u, \eta)) = \int_{\Gamma} g \sigma \cdot n \cdot t \, ds$$

Incompressible Navier–Stokes



Outflux $\approx 0.4087 \pm 10^{-4}$

Uniform

1.000.000 dofs, N hours

Adaptive

5.200 dofs, 127 seconds

```
from dolfin import *

class Noslip(SubDomain): ...

mesh = Mesh("channel-with-flap.xml.gz")
V = VectorFunctionSpace(mesh, "CG", 2)
Q = FunctionSpace(mesh, "CG", 1)
W = V*Q

# Define test functions and unknown(s)
(v, q) = TestFunctions(W)
w = Function(W)
(u, p) = split(w)

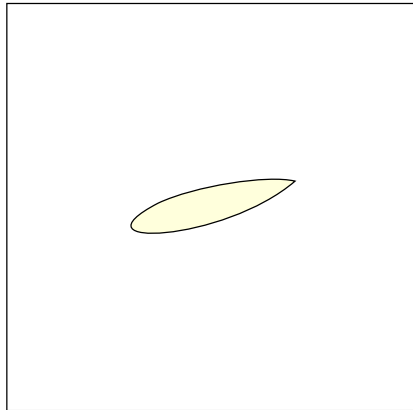
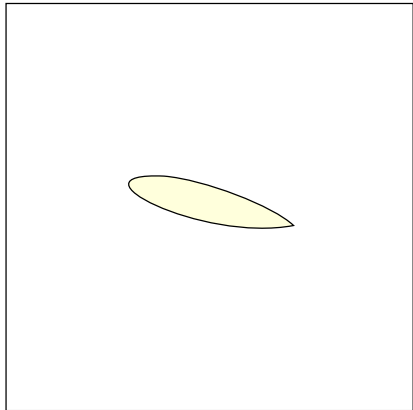
# Define (non-linear) form
n = FacetNormal(mesh)
p0 = Expression("(4.0 - x[0])/4.0")
F = (0.02*inner(grad(u), grad(v)) + inner(grad(u)*u, v)*dx
     - p*div(v) + div(u)*q + dot(v, n)*p0*ds

# Define goal functional
M = u[0]*ds(0)

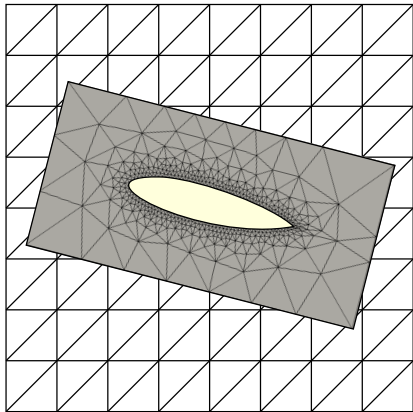
# Compute solution
tol = 1e-4
solve(F == 0, w, bcs, M, tol)
```

Cut finite elements

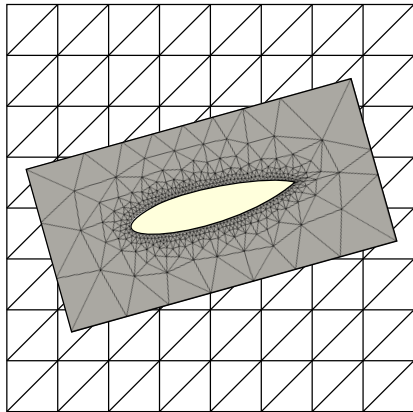
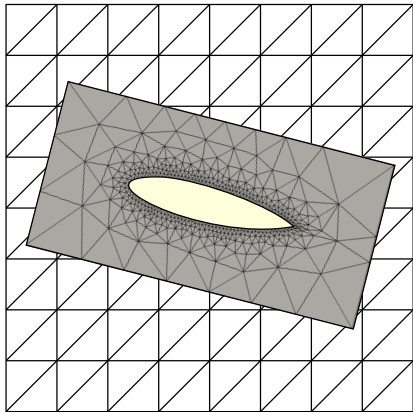
Multiple geometries – multiple meshes



Multiple geometries – multiple meshes



Multiple geometries – multiple meshes



A Nitsche formulation for the Stokes problem

Variational formulation

Find $(\mathbf{u}_h, p_h) \in V_h^k \times Q_h^l$ such that $\forall (\mathbf{v}_h, q_h) \in V_h^k \times Q_h^l$:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h) + s_h(\mathbf{u}_h, \mathbf{v}_h) - S_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = L_h(\mathbf{v}_h, q_h)$$

where

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\Omega_1 \cup \Omega_2} \underbrace{- (\langle \partial_n \mathbf{u}_h \rangle, [\mathbf{v}_h])_\Gamma}_{\text{Nitsche terms}} \\ \underbrace{- (\langle \partial_n \mathbf{v}_h \rangle, [\mathbf{u}_h])_\Gamma + \gamma (h^{-1} [\mathbf{u}_h], [\mathbf{v}_h])_\Gamma}_{\text{Nitsche terms}},$$

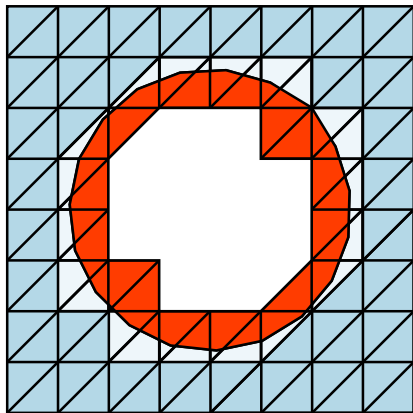
$$b_h(\mathbf{v}_h, q_h) = -(\nabla \cdot \mathbf{v}_h, q_h)_{\Omega_1 \cup \Omega_2} + \underbrace{(\mathbf{n} \cdot [\mathbf{v}_h], \langle q_h \rangle)_\Gamma}_{\text{Nitsche terms}}$$

$$s_h(\mathbf{u}_h, \mathbf{v}_h) = \underbrace{(\nabla(\mathbf{u}_{h,1} - \mathbf{u}_{h,2}), \nabla(\mathbf{v}_{h,1} - \mathbf{v}_{h,2}))_{\Omega_O}}_{\text{Ghost penalty for } u},$$

$$S_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = \delta \underbrace{\sum_{T \in \mathcal{T}_1^* \cup \mathcal{T}_2} h_T^2 (-\Delta \mathbf{u}_h + \nabla p_h, -\alpha \Delta \mathbf{v}_h + \beta \nabla q_h)_T}_{\text{Stabilization and ghost penalty}}$$

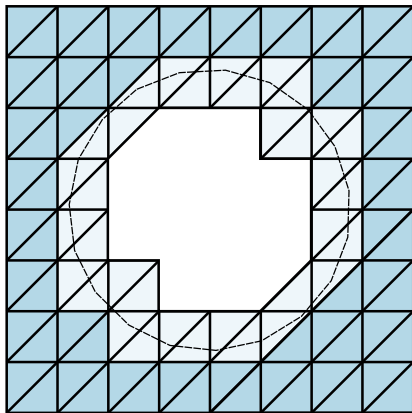
$$L_h(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) - \delta \sum_{T \in \mathcal{T}_1^* \cup \mathcal{T}_2} h_T^2 (\mathbf{f}, -\alpha \Delta \mathbf{v}_h + \beta \nabla q_h)_T.$$

Ghost-penalties added in the interface zone



$$(\nabla(\mathbf{u}_{h,1} - \mathbf{u}_{h,2}), \nabla(\mathbf{v}_{h,1} - \mathbf{v}_{h,2}))_{\Omega_O}$$

Ghost penalty for u



$$\delta \sum_{T \in \mathcal{T}_1^* \cup \mathcal{T}_2} h_T^2 (-\Delta \mathbf{u}_h + \nabla p_h, -\alpha \Delta \mathbf{v}_h + \beta \nabla q_h)_T$$

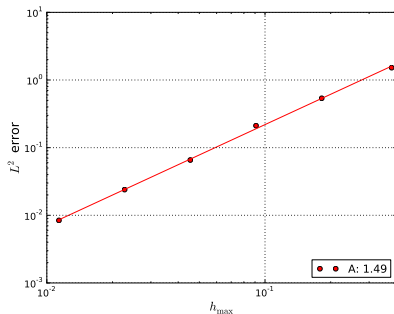
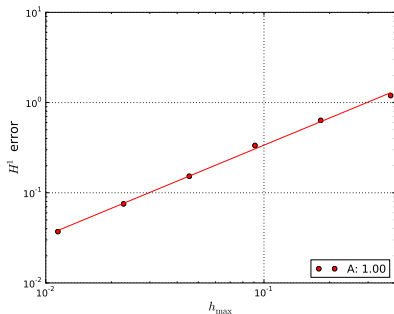
Ghost penalty for p

Optimal *a priori* estimates

Theorem

Let $k, l \geq 1$ and assume that $(\mathbf{u}, p) \in [H^{k+1}(\Omega)]^d \times H^{l+1}(\Omega)$ is a (weak) solution of the Stokes problem. Then the finite element solution $(\mathbf{u}_h, p_h) \in V_h^k \times Q_h^l$ satisfies the following error estimate:

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\| \lesssim h^k |\mathbf{u}|_{k+1} + h^{l+1} |p|_{l+1}.$$

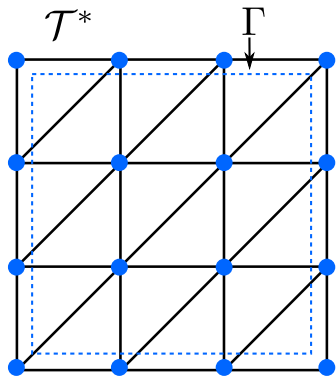


Bounded condition numbers

Theorem

There is a constant $C > 0$ *independent* of the position of Γ , s.t. the condition number of the stiffness matrix \mathcal{A} associated with the Nitsche fictitious domain method satisfies

$$\kappa(\mathcal{A}) \leq Ch^{-2},$$

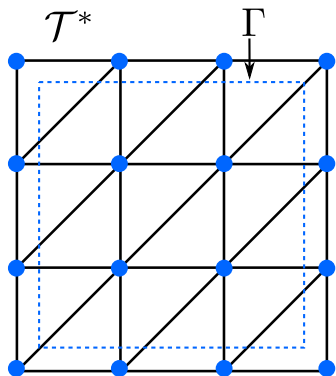


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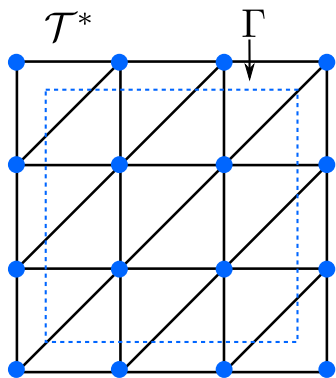


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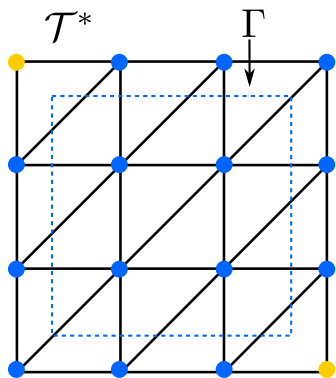


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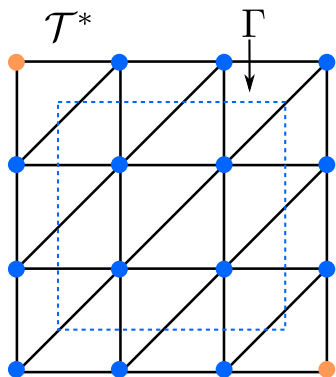


Bounded condition numbers

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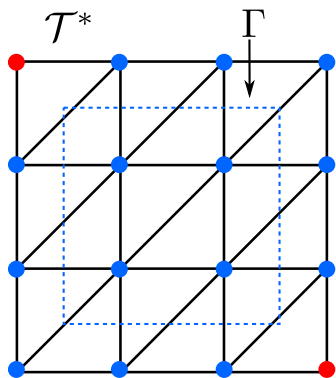


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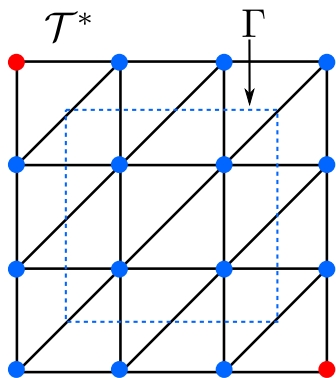


Bounded condition numbers

Theorem

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$$\kappa(\mathcal{A}) \leq Ch^{-2},$$

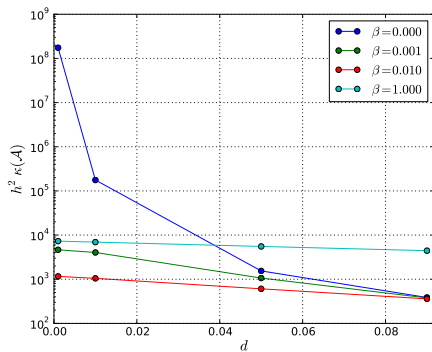
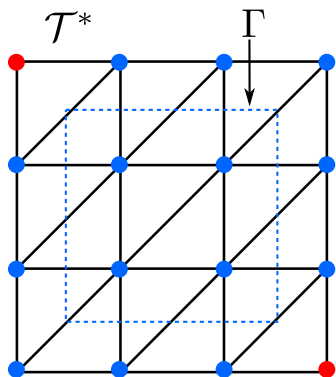


Bounded condition numbers

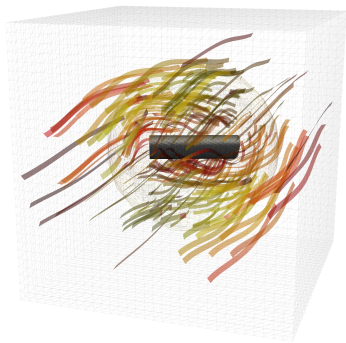
Theorem

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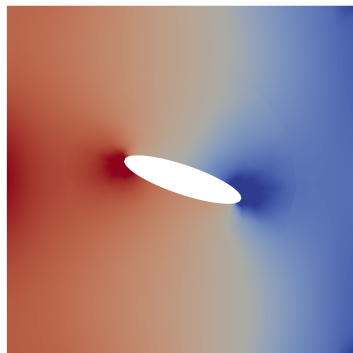
$$\kappa(\mathcal{A}) \leq Ch^{-2},$$



Stokes flow for different angles of attack

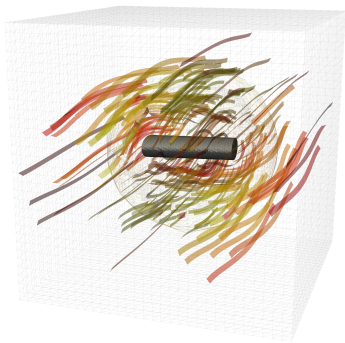


Velocity streamlines

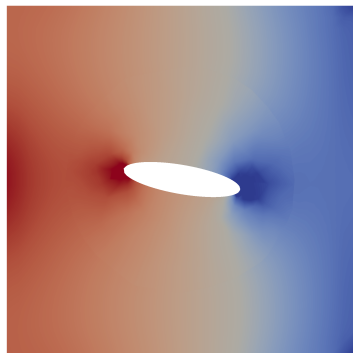


Pressure

Stokes flow for different angles of attack

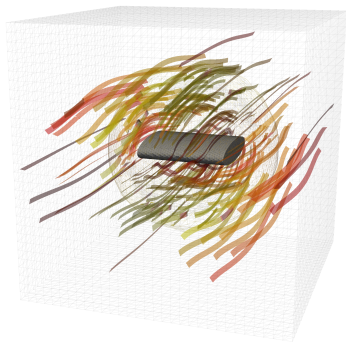


Velocity streamlines

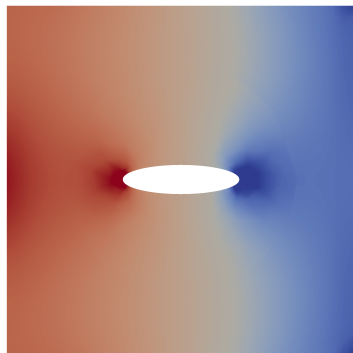


Pressure

Stokes flow for different angles of attack

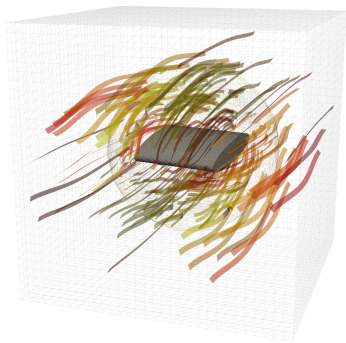


Velocity streamlines

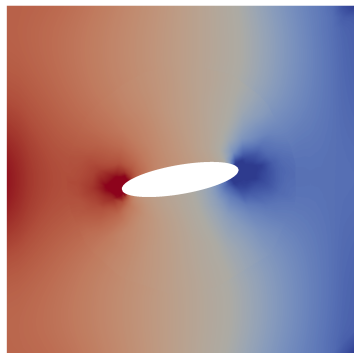


Pressure

Stokes flow for different angles of attack

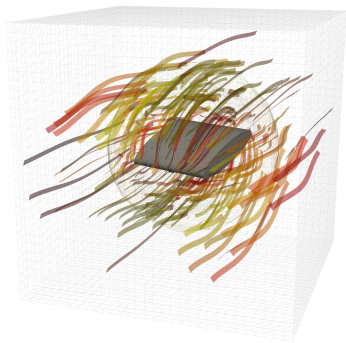


Velocity streamlines

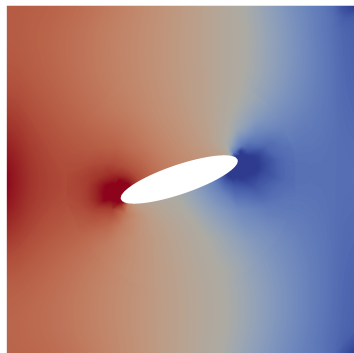


Pressure

Stokes flow for different angles of attack

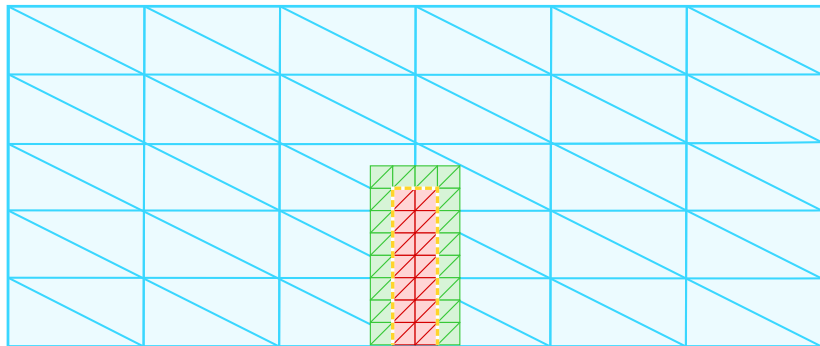


Velocity streamlines



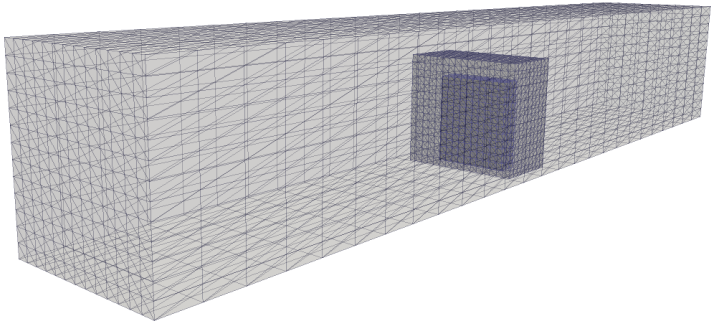
Pressure

Fluid–structure interaction on cut meshes

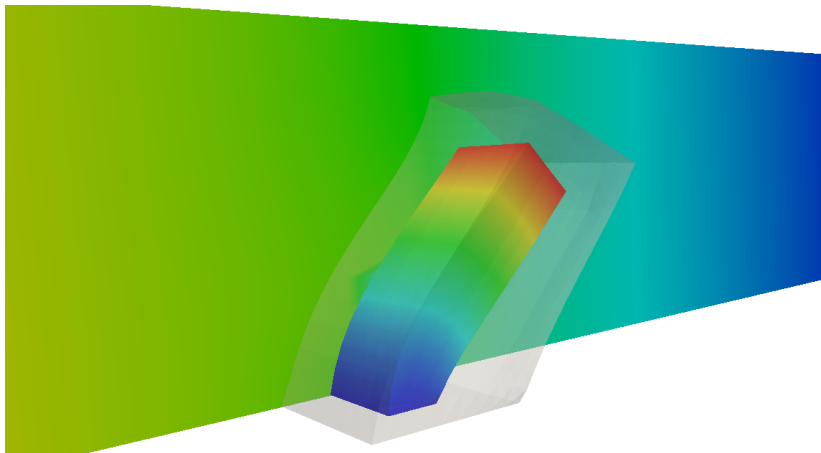


 *Fluid*  *Mesh + Fluid*  *Structure*  *Interface*

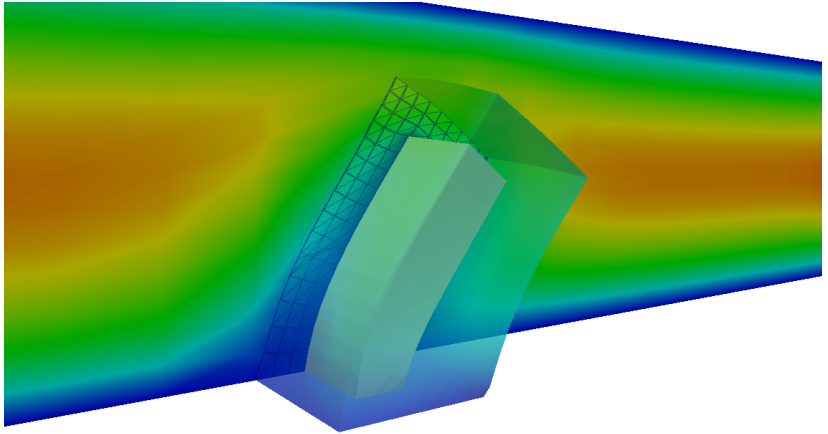
Fluid–structure interaction on cut meshes



Fluid–structure interaction: displacement

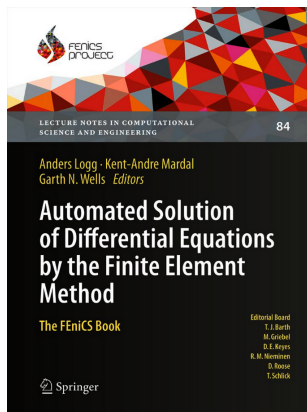


Fluid–structure interaction: velocity magnitude



Closing remarks

Ongoing activities



- Parallelization (2009)
- Automated error control (2010)
- Debian/Ubuntu (2010)
- Documentation (2011)
- FEniCS 1.0 (2011)
- The FEniCS Book (2012)

- **FEniCS'13**
Cambridge March 2013
- Visualization, mesh generation
- Parallel AMR
- Hybrid MPI/OpenMP
- Overlapping/intersecting meshes

Summary

- Automated solution of PDE
- Easy install
- Easy scripting in Python
- Efficiency by automated code generation
- Free/open-source (LGPL)

<http://fenicsproject.org/>

