The FEniCS Project

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* Credits: `http://fenicsproject.org/about/team.html`

What is FEniCS?

FEniCS is an automated programming environment for differential equations

- C++/Python library
- Initiated 2003 in Chicago
- 1000–2000 monthly downloads
- Part of Debian and Ubuntu
- Licensed under the GNU LGPL

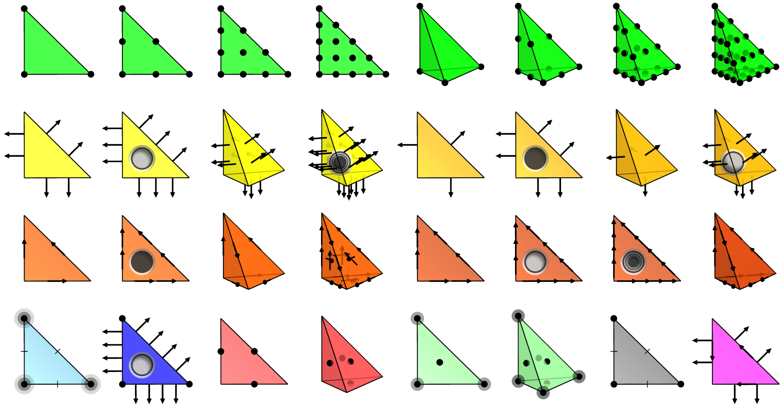
`http://fenicsproject.org/`

Collaborators

Simula Research Laboratory, University of Cambridge, University of Chicago, Texas Tech University, University of Texas at Austin, KTH Royal Institute of Technology, ...

FEniCS is automated FEM

- Automated generation of basis functions
- Automated evaluation of variational forms
- Automated finite element assembly
- Automated adaptive error control



FEniCS is automated scientific computing

Input

- $A(u) = f$
- $\epsilon > 0$

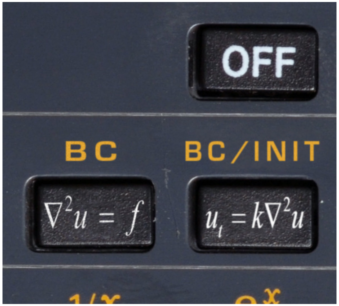
Output

- Approximate solution:

$$u_h \approx u$$

- Guaranteed accuracy:

$$\|u - u_h\| \leq \epsilon$$



How to use FEniCS?

Installation

- Official packages for Debian and Ubuntu
- Drag and drop installation on Mac OS X
- Binary installer for Windows
- Automated installation from source

Hello World in FEniCS: problem formulation

Poisson's equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Finite element formulation

Find $u \in V$ such that

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f \, v \, \mathrm{d}x}_{L(v)} \quad \forall \, v \in V$$

Hello World in FEniCS: implementation

```python
from dolfin import *

mesh = UnitSquare(32, 32)

V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("x[0]*x[1]")

a = dot(grad(u), grad(v))*dx
L = f*v*dx

bc = DirichletBC(V, 0.0, DomainBoundary())

u = Function(V)
solve(a == L, u, bc)
plot(u)
```

Linear elasticity
Differential equation

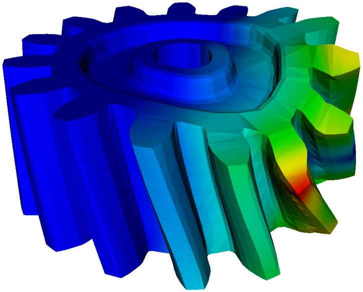
Differential equation:

$$-\nabla \cdot \sigma(u) = f$$

where

$$\sigma(v) = 2\mu\epsilon(v) + \lambda \mathrm{tr}\, \epsilon(v)\, I$$
$$\epsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^\top)$$

- Displacement $u = u(x)$
- Stress $\sigma = \sigma(x)$



Linear elasticity

Variational formulation

Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall\, v \in \hat{V}$$

where

$$\begin{aligned} a(u, v) &= \langle \sigma(u), \epsilon(v) \rangle \\ L(v) &= \langle f, v \rangle \end{aligned}$$

Linear elasticity

Implementation

```
element = VectorElement("Lagrange", "tetrahedron", 1)

v = TestFunction(element)
u = TrialFunction(element)
f = Function(element)

def epsilon(v):
    return 0.5*(grad(v) + grad(v).T)

def sigma(v):
    return 2.0*mu*epsilon(v) + lmbda*tr(epsilon(v))*I

a = inner(sigma(u), epsilon(v))*dx
L = dot(f, v)*dx
```

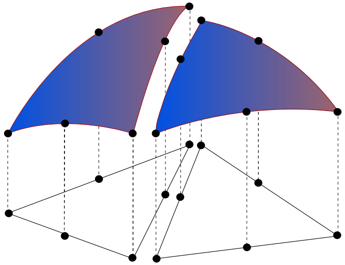
Poisson's equation with DG elements

Differential equation

Differential equation:

$$-\Delta u = f$$

- $u \in L^2$
- u discontinuous across element boundaries



Poisson's equation with DG elements

Variational formulation (interior penalty method)

Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall\, v \in V$$

where

$$\begin{aligned}
a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad + \sum_S \int_S -\langle \nabla u \rangle \cdot [\![v]\!]_n - [\![u]\!]_n \cdot \langle \nabla v \rangle + (\alpha/h)[\![u]\!]_n \cdot [\![v]\!]_n \,\mathrm{d}S \\
&\quad + \int_{\partial\Omega} -\nabla u \cdot [\![v]\!]_n - [\![u]\!]_n \cdot \nabla v + (\gamma/h) u v \,\mathrm{d}s \\
L(v) &= \int_\Omega f v \,\mathrm{d}x + \int_{\partial\Omega} g v \,\mathrm{d}s
\end{aligned}$$

Poisson's equation with DG elements

Implementation

```
V = FunctionSpace(mesh, "DG", 1)

u = TrialFunction(V)
v = TestFunction(V)

f = Expression(...)
g = Expression(...)
n = FacetNormal(mesh)
h = CellSize(mesh)

a = dot(grad(u), grad(v))*dx
  - dot(avg(grad(u)), jump(v, n))*dS
  - dot(jump(u, n), avg(grad(v)))*dS
  + alpha/avg(h)*dot(jump(u, n), jump(v, n))*dS
  - dot(grad(u), jump(v, n))*ds
  - dot(jump(u, n), grad(v))*ds
  + gamma/h*u*v*ds
```

Oelgaard, Logg, Wells, *Automated Code Generation for Discontinuous Galerkin Methods* (2008)

Simple prototyping and development in Python

```
# Tentative velocity step (sigma formulation)
U = 0.5*(u0 + u)
F1 = rho*(1/k)*inner(v, u - u0)*dx +
rho*inner(v, grad(u0)*(u0 - w))*dx \
   + inner(epsilon(v), sigma(U, p0))*dx \
   + inner(v, p0*n)*ds - mu*inner(grad(U).T*n, v)*ds \
   - inner(v, f)*dx
a1 = lhs(F1)
L1 = rhs(F1)
```

```
class StVenantKirchhoff(MaterialModel):

    def model_info(self):
        self.num_parameters = 2
        self.kinematic_measure = \
          "GreenLagrangeStrain"

    def strain_energy(self, parameters):
        E = self.E
        [mu, lmbda] = parameters
        return lmbda/2*(tr(E)**2) + mu*tr(E*E)
```

```
class GentThomas(MaterialModel):

    def model_info(self):
        self.num_parameters = 2
        self.kinematic_measure = \
          "CauchyGreenInvariants"

    def strain_energy(self, parameters):
        I1 = self.I1
        I2 = self.I2

        [C1, C2] = parameters
        return C1*(I1 - 3) + C2*ln(I2/3)
```

```
# Time-stepping loop
while True:

    # Fixed point iteration on FSI problem
    for iter in range(maxiter):

        # Solve fluid subproblem
        F.step(dt)

        # Transfer fluid stresses to structure
        Sigma_F = F.compute_fluid_stress(u_F0, u_F1,
                                         p_F0, p_F1,
                                         U_M0, U_M1)
        S.update_fluid_stress(Sigma_F)

        # Solve structure subproblem
        U_S1, P_S1 = S.step(dt)

        # Transfer structure displacement to fluidmesh
        M.update_structure_displacement(U_S1)

        # Solve mesh equation
        M.step(dt)

        # Transfer mesh displacement to fluid
        F.update_mesh_displacement(U_M1, dt)
```

```
# Fluid residual contributions
R_F0 = w*inner(EZ_F - Z_F, Dt_U_F - div(Sigma_F))*dx_F
R_F1 = avg(w)*inner(EZ_F('+') - Z_F('+'),
                    jump(Sigma_F, N_F))*dS_F
R_F2 = w*inner(EZ_F - Z_F, dot(Sigma_F, N_F))*ds
R_F3 = w*inner(EY_F - Y_F,
               div(J(U_M)*dot(inv(F(U_M)), U_F)))*dx_F
```

Simple development of specialized applications

```
# Define Cauchy stress tensor
def sigma(v,w):
    return 2.0*mu*0.5*(grad(v) + grad(v).T)  -
w*Identity(v.cell().d)

# Define symmetric gradient
def epsilon(v):
    return  0.5*(grad(v) + grad(v).T)

# Tentative velocity step (sigma formulation)
U = 0.5*(u0 + u)
F1 = rho*(1/k)*inner(v, u - u0)*dx +
rho*inner(v, grad(u0)*(u0 - w))*dx \
   + inner(epsilon(v), sigma(U, p0))*dx \
   + inner(v, p0*n)*ds - mu*inner(grad(U).T*n, v)*ds \
   - inner(v, f)*dx
a1 = lhs(F1)
L1 = rhs(F1)

# Pressure correction
a2 = inner(grad(q), k*grad(p))*dx
L2 = inner(grad(q), k*grad(p0))*dx - q*div(u1)*dx

# Velocity correction
a3 = inner(v, u)*dx
L3 = inner(v, u1)*dx + inner(v, k*grad(p0 - p1))*dx
```

- The Navier–Stokes solver is implemented in Python/FEniCS
- FEniCS allows solvers to be implemented in a minimal amount of code
- Simple integration with application specific code and data management

FEniCS under the hood

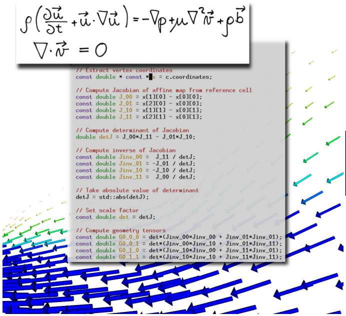
Automatic code generation

Input

Equation (variational problem)

Output

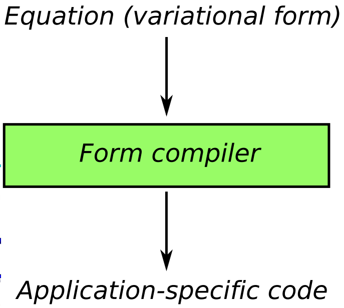
Efficient application-specific code



$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla p+\mu\nabla^2\vec{v}+\rho\vec{b}$$

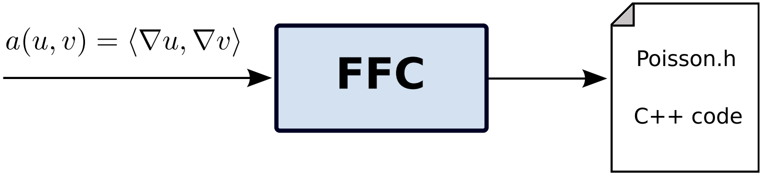
$$\nabla\cdot\vec{v}=0$$

```
// Extract vertex coordinates
const double * const *x = c.coordinates;

// Compute Jacobian of affine map from reference cell
const double J_00 = x[1][0] - x[0][0];
const double J_01 = x[2][0] - x[0][0];
const double J_10 = x[1][1] - x[0][1];
const double J_11 = x[2][1] - x[0][1];

// Compute determinant of Jacobian
double detJ = J_00*J_11 - J_01*J_10;

// Compute inverse of Jacobian
const double Jinv_00 =  J_11 / detJ;
const double Jinv_01 = -J_01 / detJ;
const double Jinv_10 = -J_10 / detJ;
const double Jinv_11 =  J_00 / detJ;

// Take absolute value of determinant
detJ = std::abs(detJ);

// Set scale factor
const double det = detJ;

// Compute geometry tensors
const double G0_0_0 = det*(Jinv_00*Jinv_00 + Jinv_01*Jinv_01);
const double G0_0_1 = det*(Jinv_00*Jinv_10 + Jinv_01*Jinv_11);
const double G0_1_0 = det*(Jinv_10*Jinv_00 + Jinv_11*Jinv_01);
const double G0_1_1 = det*(Jinv_10*Jinv_10 + Jinv_11*Jinv_11);
```



Code generation framework

- UFL - Unified Form Language
- UFC - Unified Form-assembly Code
- Form compilers: FFC, SyFi



Form compiler interfaces

Command-line

```
>> ffc poisson.ufl
```

Just-in-time

```
V = FunctionSpace(mesh, "CG", 3)
u = TrialFunction(V)
v = TestFunction(V)
A = assemble(dot(grad(u), grad(v))*dx)
```

Code generation system

```
mesh = UnitSquare(32, 32)

V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("x[0]*x[1]")

a = dot(grad(u), grad(v))*dx
L = f*v*dx

bc = DirichletBC(V, 0.0, DomainBoundary())

A = assemble(a)
b = assemble(L)
bc.apply(A, b)

u = Function(V)
solve(A, u.vector(), b)
```

Code generation system

```
mesh = UnitSquare(32, 32)

V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
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f = Expression("x[0]*x[1]")

a = dot(grad(u), grad(v))*dx
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bc.apply(A, b)

u = Function(V)
solve(A, u.vector(), b)
```
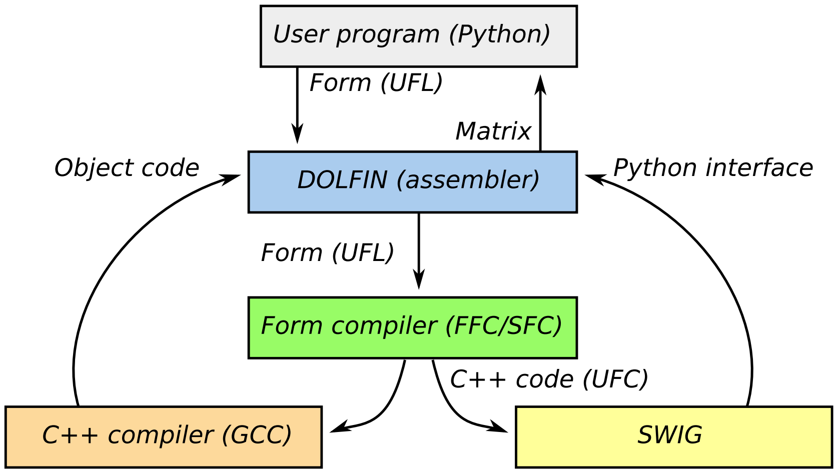

(Python, C++–SWIG–Python, Python–JIT–C++–GCC–SWIG–Python)

Just-In-Time (JIT) compilation

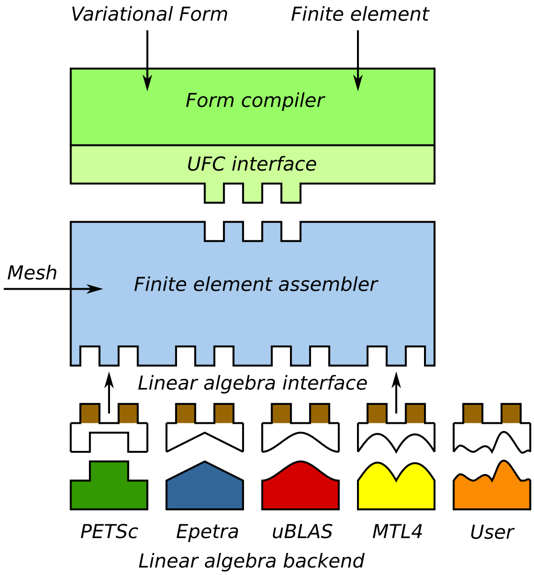


Basic API

- `Mesh`, `Vertex`, `Edge`, `Face`, `Facet`, `Cell`
- `FiniteElement`, `FunctionSpace`
- `TrialFunction`, `TestFunction`, `Function`
- `grad()`, `curl()`, `div()`, ...
- `Matrix`, `Vector`, `KrylovSolver`, `LUSolver`
- `assemble()`, `solve()`, `plot()`

- Python interface generated semi-automatically by SWIG
- C++ and Python interfaces almost identical

Assembler interfaces



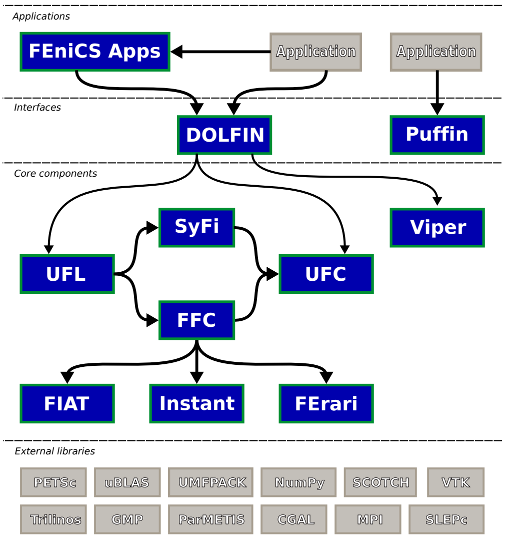
Linear algebra

- Generic linear algebra interface to
  - PETSc
  - Trilinos/Epetra
  - uBLAS
  - MTL4
- Eigenvalue problems solved by SLEPc for PETSc matrix types
- Matrix-free solvers ("virtual matrices")

Linear algebra backends

```
>>> from dolfin import *
>>> parameters["linear_algebra_backend"] = "PETSc"
>>> A = Matrix()
>>> parameters["linear_algebra_backend"] = "Epetra"
>>> B = Matrix()
```

FEniCS software components



Geometry and meshing

Geometry and meshing

Built-in meshing

```
mesh = UnitSquare(64, 64)
mesh = UnitCube(64, 64, 64)
```

External mesh generators

```
mesh = Mesh("mesh.xml")
```

```
dolfin-convert mesh.inp mesh.xml
```

Conversion from Gmsh, Medit, Tetgen, Diffpack, Abaqus, ExodusII, Star-CD

Extensions / work in progress

- Constructive solid geometry (CSG)
- Meshing from biomedical image data using VMTK

Constructive solid geometry (CSG)

Boolean operators

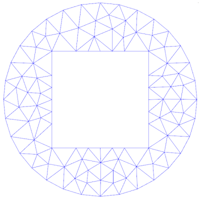
$A \cup B$	`A + B`
$A \cap B$	`A * B`
$A \setminus B$	`A - B`

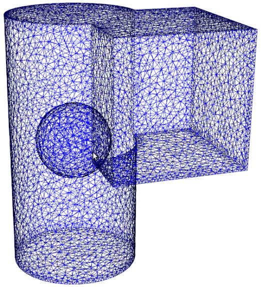
Implementation

- Modeled after NETGEN
- Implemented using CGAL

Example

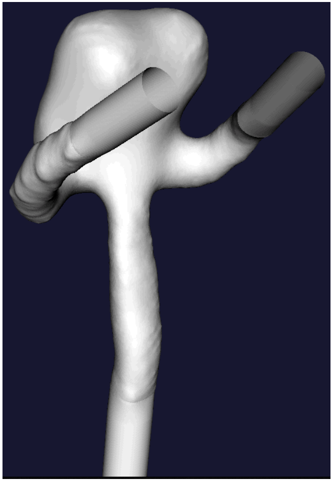
```
r = Rectangle(-1, -1, 1, 1)
c = Circle(0, 0, 1)
g = c - r
mesh = Mesh(g)
```





Meshing from biomedical images

- Biomedical image data (DICOM)
- VMTK generates high quality FEniCS meshes
- Adaptive *a priori* graded meshes
- Simple specification of boundary markers
- Resolution of boundary layers



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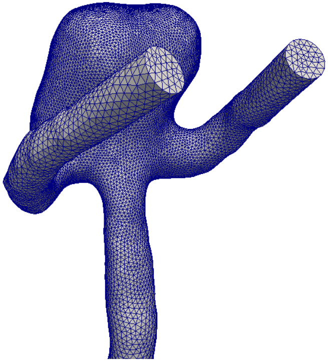

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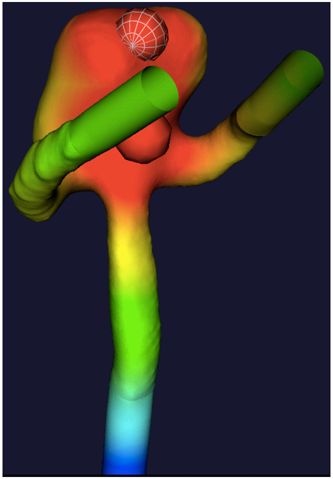

Antiga, Mardal, Valen-Sendstad, Tangui Morvan

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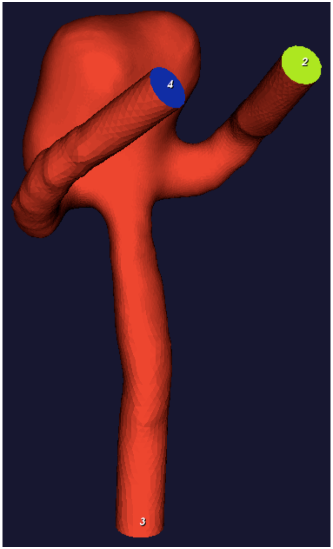

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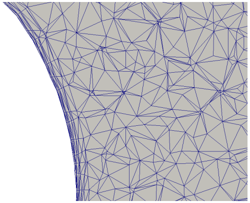


Antiga, Mardal, Valen-Sendstad, Tangui Morvan

Automated error control

Automated goal-oriented error control

Input

- Variational problem: Find $u \in V$: $a(u, v) = L(v) \quad \forall\, v \in V$
- Quantity of interest: $\mathcal{M} : V \to \mathbb{R}$
- Tolerance: $\epsilon > 0$

Objective

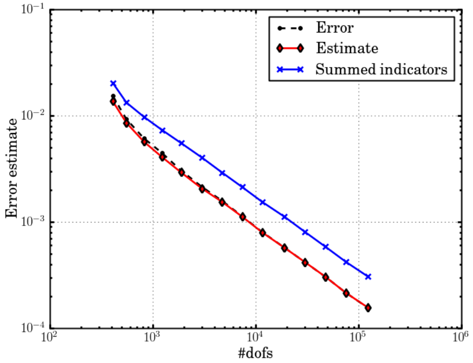
Find $V_h \subset V$ such that $|\mathcal{M}(u) - \mathcal{M}(u_h)| < \epsilon$ where

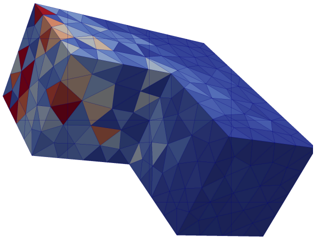
$$a(u_h, v) = L(v) \quad \forall\, v \in V_h$$

Automated in FEniCS (for linear and nonlinear PDE)

```
solve(a == L, u, M=M, tol=1e-3)
```

Poisson’s equation

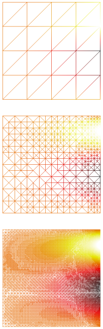


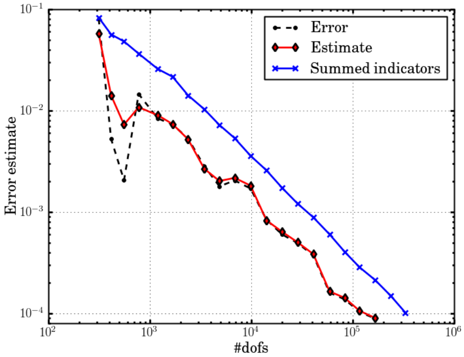


$$a(u,v) = \langle \nabla u, \nabla v \rangle$$

$$\mathcal{M}(u) = \int_\Gamma u \, \mathrm{d}s, \quad \Gamma \subset \partial\Omega$$

A three-field mixed elasticity formulation

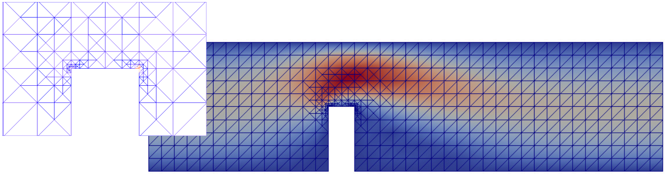




$$a((\sigma, u, \gamma), (\tau, v, \eta)) = \langle A\sigma, \tau\rangle + \langle u, \operatorname{div}\tau\rangle + \langle \operatorname{div}\sigma, v\rangle + \langle \gamma, \tau\rangle + \langle \sigma, \eta\rangle$$

$$\mathcal{M}((\sigma, u, \eta)) = \int_\Gamma g\,\sigma\cdot n\cdot t\,\mathrm{d}s$$

Incompressible Navier–Stokes



$\text{Outflux} \approx 0.4087 \pm 10^{-4}$

Uniform

1.000.000 dofs, N hours

Adaptive

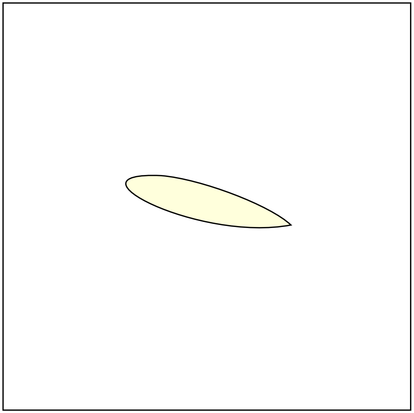
5.200 dofs, 127 seconds

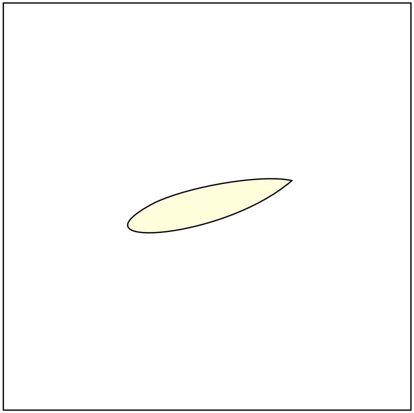
```
from dolfin import *

class Noslip(SubDomain): ...

mesh = Mesh("channel-with-flap.xml.gz")
V = VectorFunctionSpace(mesh, "CG", 2)
Q = FunctionSpace(mesh, "CG", 1)
W = V*Q

# Define test functions and unknown(s)
(v, q) = TestFunctions(W)
w = Function(W)
(u, p) = split(w)

# Define (non-linear) form
n = FacetNormal(mesh)
p0 = Expression("(4.0 - x[0])/4.0")
F = (0.02*inner(grad(u), grad(v)) + inner(grad(u)*u), v)*dx
    - p*div(v) + div(u)*q + dot(v, n)*p0*ds

# Define goal functional
M = u[0]*ds(0)

# Compute solution
tol = 1e-4
solve(F == 0, w, bcs, M, tol)
```

Rognes, Logg, *Automated Goal-Oriented Error Control I* (2010)

Cut finite elements

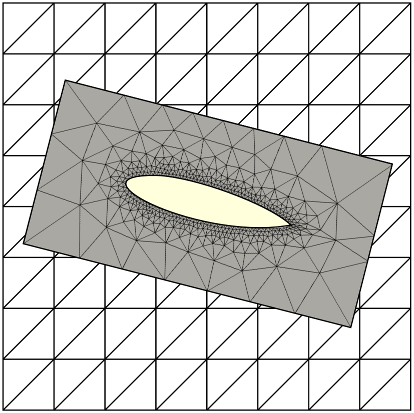
Multiple geometries – multiple meshes





Massing, Larson, Logg, Rognes, *A Nitsche overlapping mesh method for the Stokes problem* (2012)

Multiple geometries – multiple meshes



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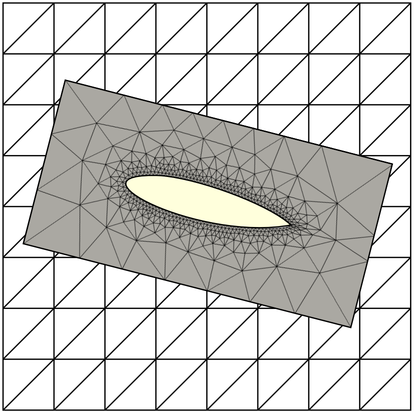
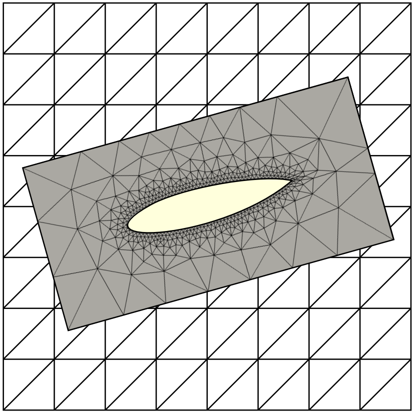
Multiple geometries – multiple meshes



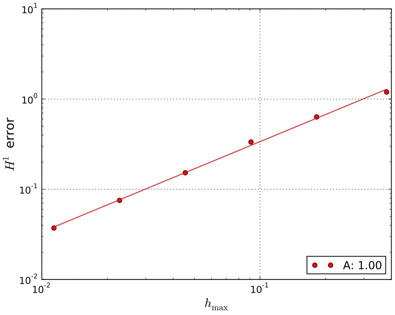


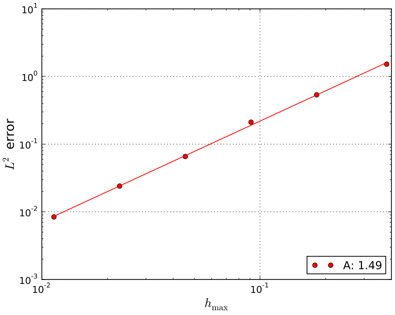
Massing, Larson, Logg, Rognes, *A Nitsche overlapping mesh method for the Stokes problem* (2012)

Optimal *a priori* estimates

Theorem
Let $k, l \geqslant 1$ and assume that $(\boldsymbol{u}, p) \in [H^{k+1}(\Omega)]^d \times H^{l+1}(\Omega)$ is a (weak) solution of the Stokes problem. Then the finite element solution $(\boldsymbol{u}_h, p_h) \in V_h^k \times Q_h^l$ satisfies the following error estimate:

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \lesssim h^k |\boldsymbol{u}|_{k+1} + h^{l+1} |p|_{l+1}.$$



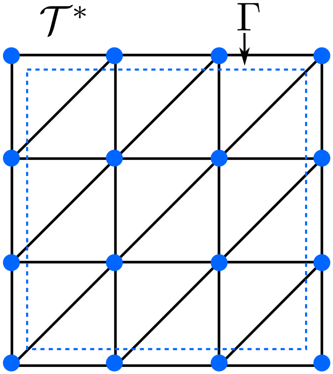


Bounded condition numbers

Theorem

There is a constant $C > 0$ independent of the position of Γ, s.t. the condition number of the stiffness matrix $\mathcal{A}$ associated with the Nitsche fictitious domain method satisfies

$$\kappa(\mathcal{A}) \leqslant Ch^{-2},$$



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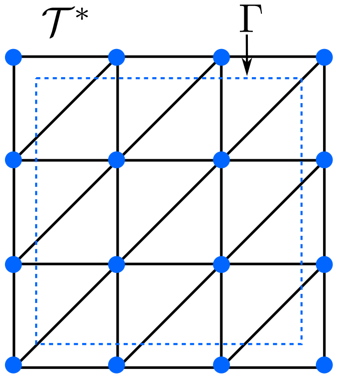

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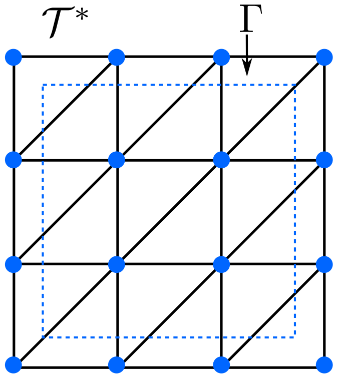

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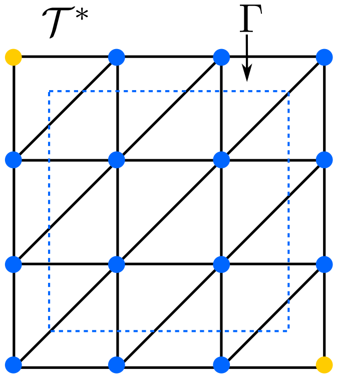

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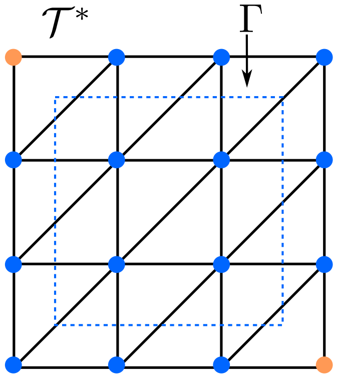

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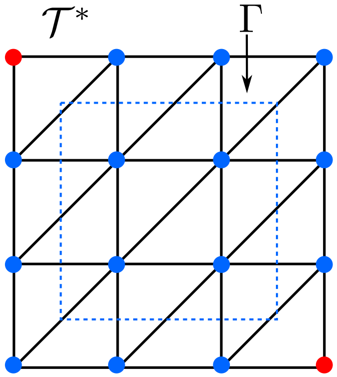

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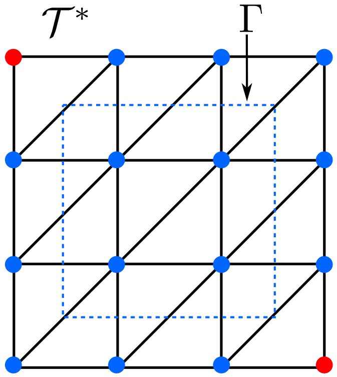

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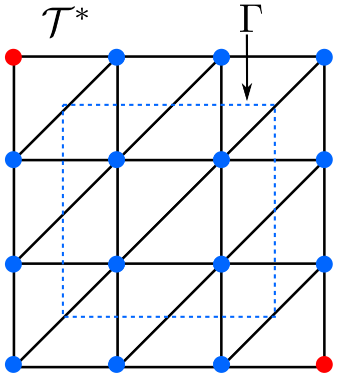
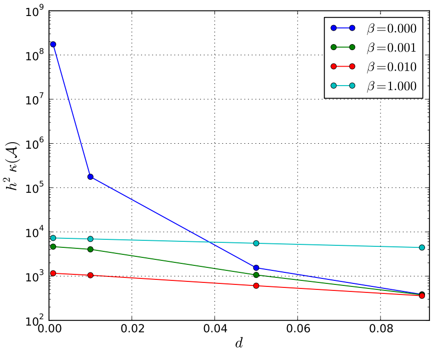
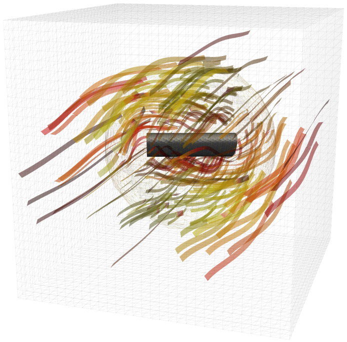
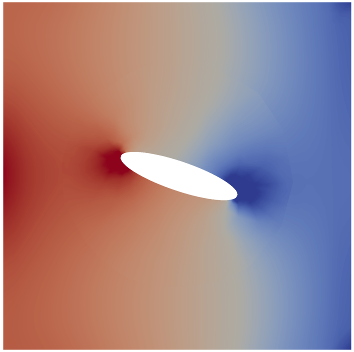




Stokes flow for different angles of attack

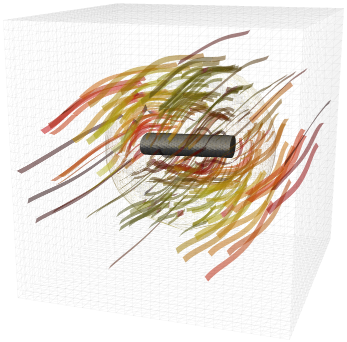


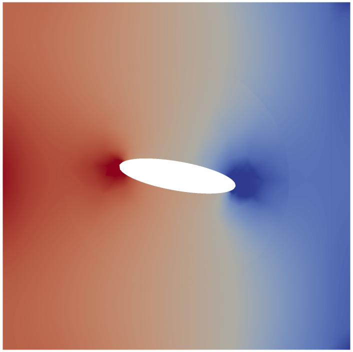


Velocity streamlines

Pressure

Stokes flow for different angles of attack





Velocity streamlines

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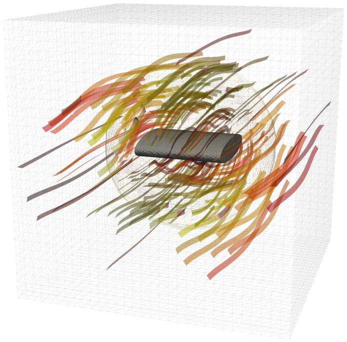
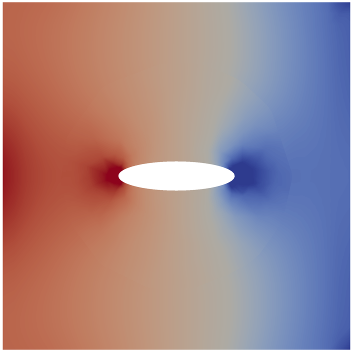
Stokes flow for different angles of attack


Velocity streamlines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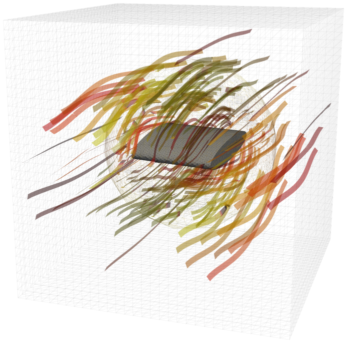
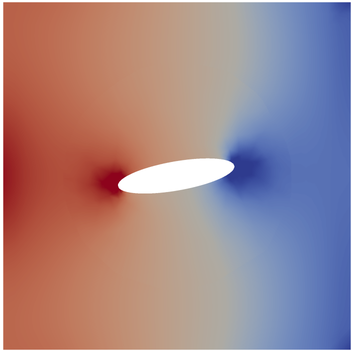


Velocity streamlines



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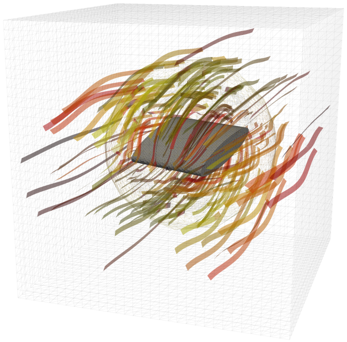
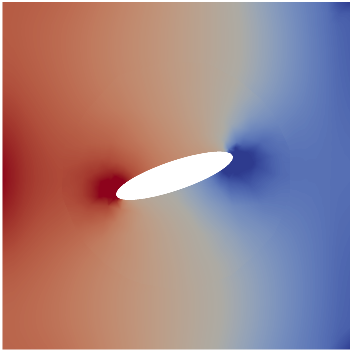
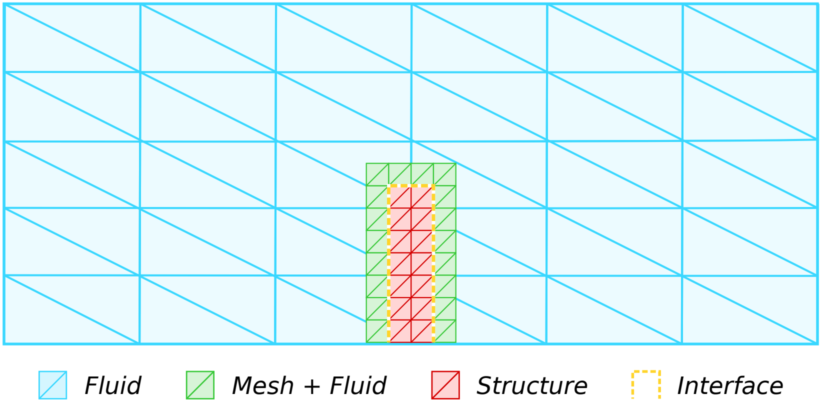

Velocity streamlines


Pressure

Fluid–structure interaction on cut meshes



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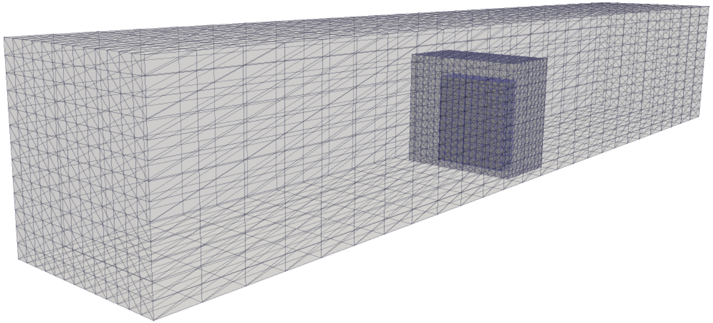
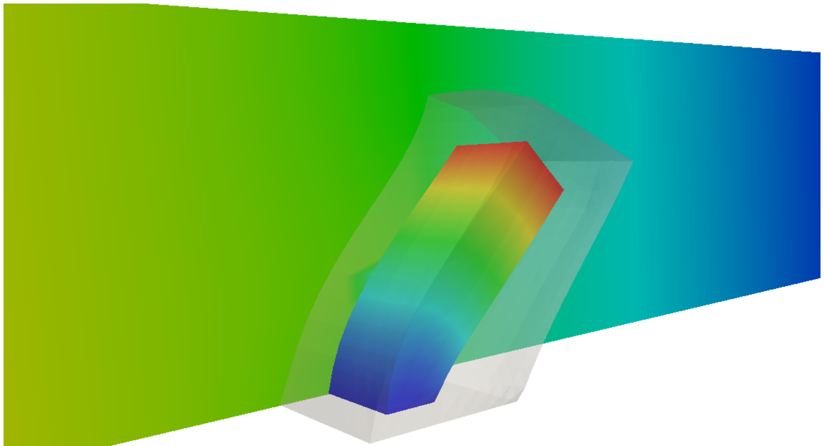


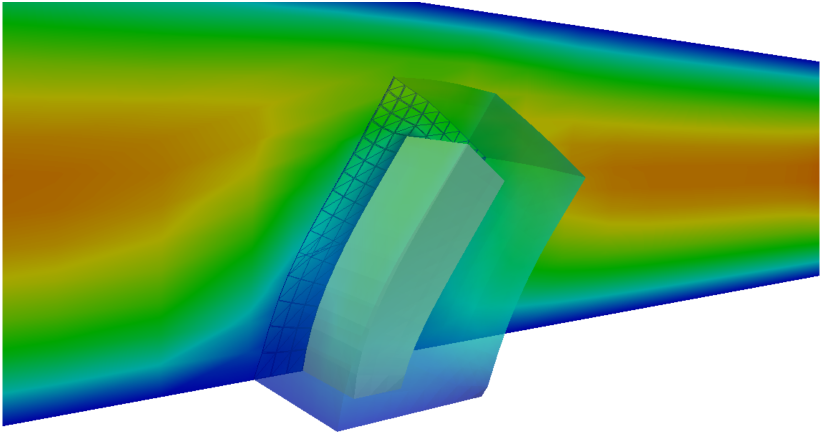
Larson, Logg, Massing, Rognes, *Fluid–structure interaction on cut meshes* (in preparation)

Fluid–structure interaction: displacement



Larson, Logg, Massing, Rognes, *Fluid–structure interaction on cut meshes* (in preparation)

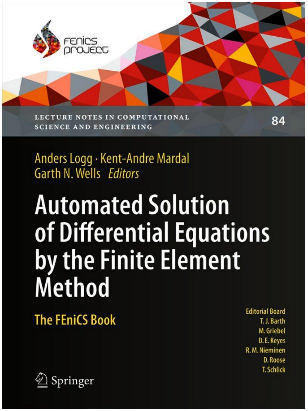
Fluid–structure interaction: velocity magnitude



Larson, Logg, Massing, Rognes, *Fluid–structure interaction on cut meshes* (in preparation)

Closing remarks

Current and future plans


FEniCS PROJECT

LECTURE NOTES IN COMPUTATIONAL SCIENCE AND ENGINEERING 84

Anders Logg · Kent-Andre Mardal
Garth N. Wells *Editors*

Automated Solution of Differential Equations by the Finite Element Method

The FEniCS Book

Editorial Board
T. J. Barth
M. Griebel
D. E. Keyes
R. M. Nieminen
D. Roose
T. Schlick

Springer

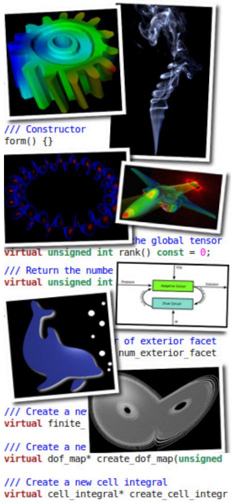
- Parallelization (2009)
- Automated error control (2010)
- Debian/Ubuntu (2010)
- Documentation (2011)
- FEniCS 1.0 (2011)
- The FEniCS Book (2012)

- **FEniCS'13**
  Cambridge March 2013
- Visualization, mesh generation
- Parallel AMR
- Hybrid MPI/OpenMP
- Overlapping/intersecting meshes

Summary

- Automated solution of PDE
- Easy install
- Easy scripting in Python
- Efficiency by automated code generation
- Free/open-source (LGPL)

`http://fenicsproject.org/`



```
/// Constructor
form() {}
he global tensor
virtual unsigned int rank() const = 0;
/// Return the numbe
virtual unsigned int
r of exterior facet
num_exterior_facet
/// Create a ne
virtual finite_
/// Create a ne
virtual dof_map* create_dof_map(unsigned
/// Create a new cell integral
virtual cell_integral* create_cell_integr
```