# FEniCS Course

## Lecture 16: Optimal control of the Navier-Stokes equations

*Contributors*
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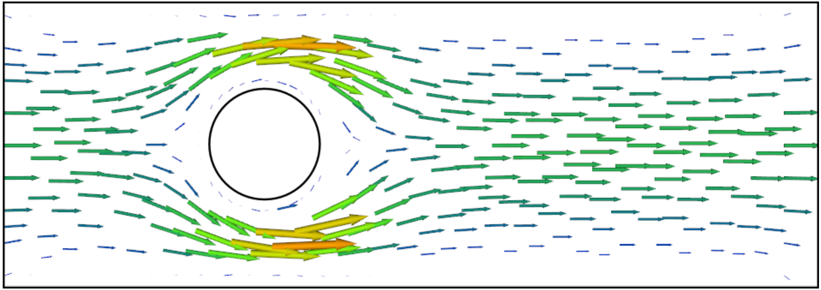
# The FEniCS challenge!

Consider steady flow around a cylinder driven by a pressure difference at the left and right boundaries:



- Imagine you can place sponges in the top half (light green) area of the domain.
- How would you place the sponges in order to minimise dissipation of the flow into heat?

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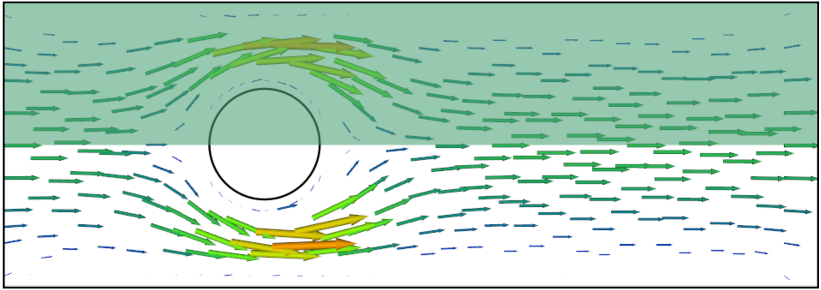

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$$\min_{u,f} \int_\Omega \langle \nabla u, \nabla u \rangle \, \mathrm{d}x + \alpha \int_\Omega \langle f, f \rangle \, \mathrm{d}x$$

subject to:

$$\begin{aligned} -\nu \Delta u + \nabla u \cdot u - \nabla p &= -f u \qquad && \text{in } \Omega, \\ \operatorname{div}(u) &= 0 && \text{in } \Omega, \end{aligned}$$

with:

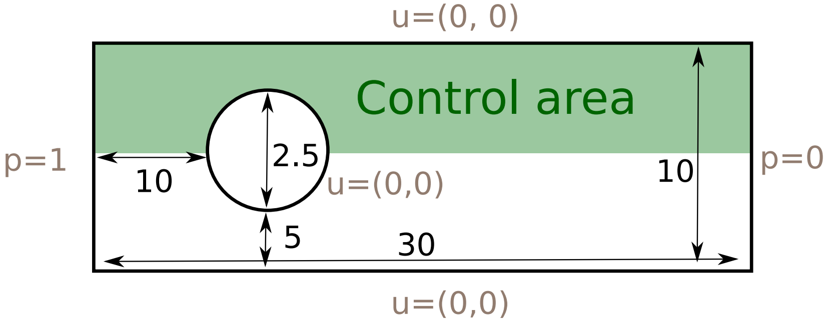
- $u$ the velocity,
- $p$ the pressure,
- $f$ the control function.
- $\nu$ the viscosity,
- $\alpha$ the regularisation parameter,

The boundary conditions are:

- $p = 1$ on the left,
- $p = 0$ on the right,
- $u = (0, 0)$ on the top and bottom and circle.

# The FEniCS challenge!

## Domain




1. Write a new program and generate a mesh with the above domain.

# The FEniCS challenge!

## Navier-Stokes solver

The variational formulation of the Navier-Stokes equations is:
Find $u, p \in V \times Q$ such that:

$$\int_\Omega \nu \nabla u \cdot \nabla v + (\nabla u \cdot u + f u) \cdot v + p \,\mathrm{div}\, v \,\mathrm{d}x = -\int_{\partial\Omega} p_0 v \cdot n \,\mathrm{d}s,$$

$$\int_\Omega \mathrm{div}\, u \, q \,\mathrm{d}x = 0$$

for all $v \in V$ and all $q \in Q$.

2. Create a `MixedFunctionSpace` consisting of continuous piecewise quadratic vector fields for the velocity and continuous piecewise linears for the pressure.
3. Solve the Navier-Stokes equation for $f = 0$ and the given boundary conditions.

# The FEniCS challenge!

## Optimise

3. Import `dolfin-adjoint` and define the control parameter and functional.
4. Minimise the functional and plot the optimal control.
5. What is the minimised functional?

### Note

Multiply the $\int_\Omega f u \cdot v \, \mathrm{d}x$ term in the variational formulation with the indicator function:

*Python code*

```
chi = conditional(triangle.x[1] >= 5, 1, 0)
```

to ensure that the control is only active in the top half of the domain.